Math 678
Lecture 25.

\[ \begin{aligned}
    \begin{cases}
    \mathcal{L}(u, x) = 0 \quad &\text{in} \quad \mathbb{R}^n \times (0, \infty) \\
    u = g \quad &\text{on} \quad \mathbb{R}^n \times \{ t = 0 \}.
    \end{cases}
\end{aligned} \]

CE: \quad \begin{cases}
    \dot{p}(s) = -D_x H \\
    \dot{x}(s) = D_p H \cdot p - H
    \end{cases} \quad \Rightarrow \quad \begin{cases}
    \dot{x}(s) = D_p H \\
    \dot{p}(s) = -D_x H
    \end{cases}

[\text{Hamilton eqn:}]

\[ I[w(\cdot)] := \int_0^T L(x(s), v(s)) \, ds \]

Variational formulation: \quad \min_{w \in \mathcal{A}} I[w]

\[ \begin{aligned}
    y: [0, t] &\rightarrow \mathbb{R}^n \quad y(0) = y(t) = 0 \quad \text{smooth curve} \\
    W(\cdot) &= x(\cdot) + \tau y(\cdot) \quad \tau \in \mathbb{R} \\
    W(0) &= x(0) + \tau y(0) = y + \tau \cdot 0 = y \\
    W(t) &= x(t) + \tau y(t) = x + \tau \cdot 0 = x \\
    \Rightarrow & \quad I[x] \leq I[w] \quad \text{since} \quad x = \arg\min_{w \in \mathcal{A}} I[w]
    \end{aligned} \]

\[ u^*(\cdot) := I[x(\cdot) + \tau y(\cdot)] \quad \text{has a min at} \quad \tau = 0 \]

\[ u^*(\tau) = I[x(\cdot) + \tau y(\cdot)] \quad \text{has a min at} \quad \tau = 0. \]

\[ \begin{aligned}
    u'(\tau) &= 0 \quad \text{when} \quad \tau = 0. \\
    u'(\tau) &= \mathcal{L}(x(s) + \tau y(s), x(s) + \tau y(s)) \, ds \\
    &= \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \mathcal{L}(x(s), x(s)) + \frac{\partial}{\partial y_i} \mathcal{L}(x(s), x(s)) \right) y_i \, ds \\
    &= \sum_{i=1}^{n} \int_0^T \left( \frac{\partial}{\partial x_i} \mathcal{L}(x(s), x(s)) + \frac{\partial}{\partial y_i} \mathcal{L}(x(s), x(s)) \right) y_i \, ds \\
    &= \sum_{i=1}^{n} \int_0^T \frac{\partial}{\partial x_i} \mathcal{L}(x(s), x(s)) \, ds
    \end{aligned} \]

\[ 0 = u'(0) = \sum_{i=1}^{n} \int_0^T \frac{\partial}{\partial x_i} \mathcal{L}(x(s), x(s)) \, ds \quad \Rightarrow \quad \text{EL eqns.} \]
EL eqns characterize orbit. pts., so
not only minima of $I[u]$ satisfy these
eqns. This is a system of $n$ 2nd order ODEs.

\[
\begin{align*}
\text{(2nd order ODE system)} & \quad \rightarrow \quad \text{(1st order ODE system)} \\
\text{EL system} & \quad \rightarrow \quad \text{Hamilton system}
\end{align*}
\]

Introduce variables $p(t) := D_v L(x(t), x'(t))$

Assume: this makes a solvable eqn on $v$

\[p = D_v L(v, x) \Rightarrow v = V(p, x) \quad \forall p, x \in \mathbb{R}^n\]

Def. $H(p, x) := p \cdot V(p, x) - L(V(p, x), x)$

Hamiltnian:

\[
\begin{align*}
H_x_i & = \sum_{k=1}^n p_k V_{x_i} (p, x) - L_{x_i} \cdot v = -L_{x_i} (v, x) \\
H_{p_i} (p, x) & = v_i (p, x) + \sum_{k=1}^n (p_k V_{p_i} - L_{v_k} \cdot v_k) \\
& = v_i (p, x)
\end{align*}
\]

In case of $L = \frac{1}{2} m |v|^2 - V(x)$:

\[
p = l_v = \frac{p}{m} v \quad \Rightarrow \quad v = \frac{p}{m} = V(p, x)
\]

\[
H = p \cdot \frac{p}{m} - \frac{1}{2} m |v|^2 + V(x) = \frac{1}{2m} |p|^2 + \frac{V(x)}{\text{kinetic}} + \frac{V(x)}{\text{potential}}
\]

Derivation of Hamilton eqns:

\[
\begin{align*}
W \quad \{ & H_{p_i} (p, x) = \dot{x}_i (s) \\
H_{x_i} (p, x) = -\dot{p}_i (s) \}
\end{align*}
\]

Hamilton equations $2n$ eqns

\[
\frac{d}{ds} H(p, x) = \sum_{i=1}^n (H_{p_i} \cdot \dot{p}_i + H_{x_i} \cdot \dot{x}_i) = \sum (H_{p_i} H_{x_i} + H_{x_i} H_{p_i}) = 0
\]

\[H = \text{const}\]
From now on: \( H = H(p) \) so \( H_x = 0 \).

(1) \( L: \mathbb{R}^n \rightarrow \mathbb{R} \) convex \( \implies \lim_{v \rightarrow \infty} \frac{L(v)}{|v|} = +\infty \)

(2) \( \lim_{v \rightarrow \infty} \frac{L(v)}{|v|} = +\infty \)

Thus: (Convex duality of Hamiltonian & Lagrangian)

If \( L \) satisfies (1) - (2), take \( H = L^* \) then

(1) \( H(p) \) is convex and \( \lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = \infty \)

(2) \( L = H^* \)

Corollary: TFAE: \( \left\{ \begin{array}{l}
p \cdot v = L(v) + H(p) \\
p = DL(v) \\
v = DH(p)
\end{array} \right. \)

Go back to Hamilton–Jacobi: \( \{ \dot{u} + H(Du) = 0 \} \)

CE: \( \left\{ \begin{array}{l}
\dot{p} = 0 \\
\dot{x} = DH(p) \\
\dot{z} = DH(p) \cdot p - H(p)
\end{array} \right. \)

When smooth soln exists (locally) we have

\( \dot{z} = DH(p) \cdot p - H(p) = \dot{x} \cdot p - H(p) = L(\dot{x}) \Rightarrow \\
\dot{u}(x, t) = \int L(\dot{x}(s)) \, ds + g(x(0)) \)

Wanted: modification of this formula that works even when soln becomes non-smooth.

Change action functional: \( \min_{w} \int_{t_0}^{t} L(w(s)) \, ds + g(w(0)) \) \( \text{initial data} \)

Define: \( u(x, t) := \inf_{w \in \mathcal{C}} \left\{ \int_{t_0}^{t} L(w(s)) \, ds + g(w(0)) \mid w(t) = x \right\} \)

Hopf–Lax formula: \( u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ t L(\frac{x-y}{t}) + g(y) \right\} \)