Math 678.
Lecture 24.

Compatibility conditions: $\begin{cases} z^0 = g(x^0) \\ p_i^0 = g_{x_i}(x^0) \\ F(p^0, z^0, x^0) = 0 \end{cases}$

(p^0, z^0, x^0) - <u>admissible triple</u>

$\left.\begin{array}{l} x(0) = x^0 \\ z(0) = z^0 \\ p(0) = p^0 \end{array}\right\}$ appropriate BC for char. ODE

Wanted: to perturb (p^0, z^0, x^0) s.t. compatibility conditions are still satisfied.

$y = (y_1, \dots y_{n-1}, 0) \in \Gamma \quad p(0) = q(y)$

and we want $(q(y), g(y), y)$ to be admissible

$\Rightarrow \begin{cases} q^i(y) = g_{x_i}(y) \\ F(q(y), g(y), y) = 0 \end{cases} \quad \forall y \in \Gamma$ close to x^0.

<u>Claim.</u> $\exists!$ solution to the system $\begin{cases} q(x^0) = p^0 \\ q^i(y) = g_{x_i}(y) \\ F(q(y), g(y), y) = 0 \end{cases}$

for all $y \in \Gamma$ close to x^0

as long as $F_{p_n}(p^0, z^0, x^0) \neq 0$.

(by Inverse Function Thm)

if this holds we say (p^0, z^0, x^0) is a <u>noncharacteristic pt</u>

If the Γ is not flat, this condition becomes $D_p F(p^0, z^0, x^0) \cdot \nu(x^0) \neq 0$

outward unit normal to ∂U at x^0.

Look at a local soln.

$p(s) = p(y_1, \dots y_{n-1}, s)$

$z(s) = z(y_1, \dots y_{n-1}, s)$

$x(s) = x(y_1, \dots y_{n-1}, s)$

Lemma. If $F_{p_n}(p^0, z^0, x^0) \neq 0$ (nonchar. pt) then there exists an open interval $I \subset \mathbb{R}$ containing 0, $\exists\, W_{x^0} \subset \Gamma \subset \mathbb{R}^{n-1}$, $\exists\, V_{x^0} \subset \mathbb{R}^n$ s.t. $\forall x \in V_{x^0}$ $\exists!\, s \in I$, $y \in W_{x^0}$ s.t.

$$x = X(y,s)$$

$\uparrow$ C^2-mapping

Pf: $\mathbb{X}(x^0, 0) = x^0$

As long as $\det D\mathbb{X}(x^0,0) \neq 0 \Rightarrow$ by IFT sol. of $x = X(y,s)$ exist.

$\mathbb{X}(x^0,0) = x^0 \Rightarrow \mathbb{X}(y,0) = (y,0)$

$$\begin{cases} x^j_{y_i}(x^0,0) = \begin{cases} \delta_{ij} & j = 1, \dots n-1 \\ 0, & j = n \end{cases} \\ x^j_s(x^0,0) = F_{p_j}(p^0, z^0, x^0) \end{cases}$$

$$D\mathbb{X}(x^0,0) = \left(\begin{array}{ccc|c} 1 & & 0 & F_{p_1} \\ & \ddots & & \vdots \\ 0 & & 1 & F_{p_{n-1}} \\ \hline & 0 & & F_{p_n} \end{array}\right) \Rightarrow |DX| \neq 0 \Leftrightarrow F_{p_n} \neq 0.$$

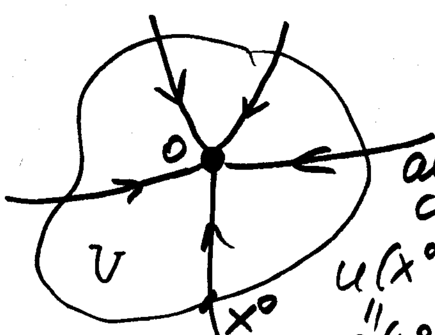
Thm (Local Existence)

$u(x) := z(y(x), s(x))$

$p(x) := p(y(x), s(x))$

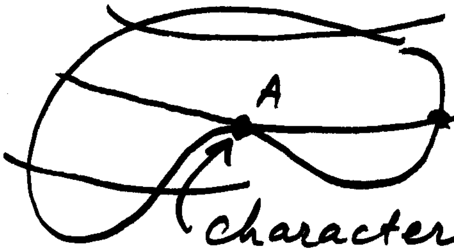
$\Rightarrow u \in C^2$ and solves the system $\begin{cases} F(Du, u, x) = 0 & \text{in } V \\ u = g & \text{on } \Gamma \end{cases}$

$F = b(x) \cdot Du(x) + c(x)u(x)$ — linear homogeneous case



along each characteristic $u(x^0) \equiv u(x)$, $\quad u(x^0) = g(x^0)$ $\Rightarrow$ only local soln close to Γ is available

at 0, char. eqns are not solvable



nonchar. pt (local solution $\exists$)

characteristic soln will be discont. ~~at long~~ unless

Hamilton-Jacobi equations:

$$\begin{cases} u_t + H(Du, x) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$$G(Du, u_t, u, x, t) = u_t + H(Du, x) = 0$$

$$Du = D_x u \quad , \quad q = (p, p_{n+1}) \quad , \quad y = (x, t)$$

$$G(q, z, y) = p_{n+1} + H(p, x) = 0.$$

$$D_q G = (D_p H(p,x), 1) \quad , \quad D_y G = (D_x H(p,x), 0)$$

$$D_z G = 0.$$

$\Rightarrow$ Char. eqns:
$$\begin{cases} \dot{x}_i(s) = H_{p_i}(p,x) \quad , \quad i = 1, \dots n \\ \dot{x}_{n+1}(s) = 1 \\ \dot{p}_i(s) = -H_{x_i}(p,x) \quad i = 1, \dots n \\ \dot{p}_{n+1}(s) = 0 \\ \dot{z}(s) = D_p H \cdot p + p_{n+1} = \\ \qquad = D_p H \cdot p - H(p,x) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{p}(s) = -D_x H(p,x) \\ \dot{z}(s) = D_p H \cdot p - H \\ \dot{x}(s) = D_p H \end{cases} \Rightarrow \begin{cases} \dot{x} = D_p H \\ \dot{p} = -D_x H \end{cases}$$

Hamilton eqns

Can show smooth solns of this system will not be extendable to all $t > 0$.

Wanted: a weak soln valid for all $t > 0$. even after char. method fails.

Calculus of variation.

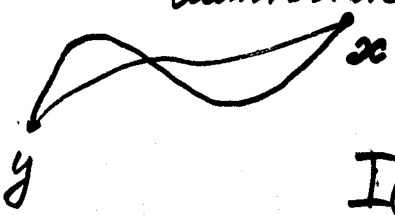
$$L(v,x) = L(v_1, \dots v_n, x_1, \dots x_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

Lagrangian $\quad DL = (D_v L, D_x L)$

Fix $x, y \in \mathbb{R}^n$, $t > 0$

Action : $I[w(\cdot)] = \int_0^t L(\dot{w}(s), w(s))\, ds$

$w \in \mathcal{A}$ $\quad \underset{\text{admissible set}}{\mathcal{A}} := \{ w(\cdot) \in C^2([0,t]; \mathbb{R}^n) \mid w(0) = y,\ w(t) = x \}$



To find: $\mathbf{X}(\cdot) = \underset{w(\cdot) \in \mathcal{A}}{\operatorname{argmin}} I[w(\cdot)]$

$$I(\mathbf{X}(\cdot)) = \min_{w(\cdot) \in \mathcal{A}} I[w(\cdot)]$$

Theorem: (Euler-Lagrange Thm)

$\mathbf{X}(\cdot)$ must satisfy Euler-Lagrange eqns:

$$-\frac{d}{ds}\left(D_v L(\dot{\mathbf{X}}(s), \mathbf{X}(s))\right) + D_x L(\dot{\mathbf{X}}(s), \mathbf{X}(s)) = 0$$

all $0 \le s \le t$

Example. $L(v,x) = \frac{1}{2} m |v|^2 - \varphi(x)$

$D_v L = m v \quad \Rightarrow \frac{d}{ds}(m v) = m \ddot{x}$

$D_x L = -\varphi'(x)$

$\Longrightarrow\ -m\ddot{x} - \varphi'(x) = 0 \quad \Rightarrow \quad m\ddot{x} = -\varphi'(x) = f(x)$

$m\ddot{x} = F$ - force

Newton's law for a mass m in a force field $f = -\varphi'(x)$ generated by $\varphi(x)$.