Math 678.
Lecture 23.

$$\begin{cases} F(Du, u, x) = 0 & \text{in } V \\ u = g & \text{on } \Gamma \end{cases}$$

Characteristic ODEs:
$$\begin{cases} \dot{p}(s) = -D_x F(p, z, x) - D_z F(p, z, x) p(s) \\ \dot{z}(s) = D_p F(p, z, x) \cdot p(s) \\ \dot{x}(s) = D_p F(p, z, x) \end{cases}$$

Ex. 1 Semilinear
$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } V = \{x_2 > 0\} \\ u = g & \text{on } \Gamma = \{x_2 = 0\} \end{cases}$$

$$F(p, z, x) = p_1 + p_2 - z^2 = 0$$

$x = (x_1, x_2)$ $\quad \dot{x}(s) = D_p F = (1, 1) \Rightarrow \begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = 1 \\ \dot{z} = p_1 + p_2 = z^2 \end{cases}$

$x(0) = (x_1^0, x_2^0)$, $x_2^0 = 0$

$\Rightarrow$ $x_1 = x_1^0 + s$ $\qquad z(0) = g(\cancel{z}(0)) = z^0$

$x_2 = s$

$\dot{z} = z^2 \Rightarrow -z^{-1} = s + C$

$\dfrac{dz}{z^2} = ds \qquad z = \dfrac{-1}{s+C} \Rightarrow z(0) = \dfrac{-1}{C} = z^0$

$\to C = -\dfrac{1}{z^0}$

$$z(s) = \frac{-1}{s - \frac{1}{z^0}} = \underline{\frac{z^0}{1 - s z^0}}, \quad \text{since } s = x_2$$

$x_1^0 = x_1 - s = x_1 - x_2$

$$\Rightarrow u(x) = u(x_1(s), x_2(s)) = z(s) = \frac{g(x_1^0)}{1 - x_2 g(x_1^0)} = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}$$

Ex. 2 Fully nonlinear
$$\begin{cases} u_{x_1} u_{x_2} = u & \text{in } V = \{x_1 > 0\} \\ u = x_2^2 & \text{on } \Gamma = \{x_1 = 0\} \end{cases}$$

$$F(p,z,x) = p_1 p_2 - z = 0, \quad D_pF = (p_2, p_1) \quad D_zF = -1$$

$$x = (x_1, x_2) \Rightarrow \begin{cases} \dot{x}_1 = p_2 & x(0) = \cancel{(x_1^0, x_2^0)}\ (0, x^0) \\ \dot{x}_2 = p_1 & \\ \dot{z} = 2p_1 p_2 & z(0) = (x^0)^2 \\ \dot{p}_1 = p_1 & p(0) = (p_1^0, p_2^0) \\ \dot{p}_2 = p_2 & \end{cases}$$

$$\dot{p} = -D_xF - D_zF\,p \Rightarrow$$

$$\Rightarrow \begin{cases} p_1 = p_1^0 e^s \\ p_2 = p_2^0 e^s \end{cases} \Rightarrow$$

$$\dot{x}_1 = p_2^0 e^s \Rightarrow$$

$$x_1 = p_2^0 e^s + C \quad x_1(0) = 0 \quad C = -p_2^0$$

$$\boxed{x_1 = p_2^0(e^s - 1)}$$

$$x_2 = p_1^0 e^s + C, \quad x_2(0) = x^0 \Rightarrow C = x^0 - p_1^0$$

$$\boxed{x_2 = x^0 + p_1^0(e^s - 1)}$$

$$\dot{z} = 2p_1^0 p_2^0 e^{2s}$$

$$z = p_1^0 p_2^0 e^{2s} + z(0) - p_1^0 p_2^0 = \boxed{x_0^2 + p_1^0 p_2^0 (e^{2s} - 1)} = \underline{= x_0^2 \cdot e^{2s}}$$

need to resolve

~~Set~~ Relations:

1) $p_1^0 p_2^0 - z(0) = 0$

$p_1^0 p_2^0 - (x^0)^2 = 0 \Rightarrow 2p_1^0 x^0 - (x^0)^2 = 0$

2) $p_2^0 = u_{x_2}(0, x^0) = 2x^0, \; x^0 \neq 0 \Rightarrow p_1^0 = \frac{x^0}{2}$

3) $\begin{cases} x_1 = 2x^0(e^s - 1) \\ x_2 = x^0 + \frac{x^0}{2}(e^s - 1) \end{cases} \qquad \begin{cases} x_1 = 2x^0(e^s - 1) \\ 4x_2 = 4x^0 + 2x^0(e^s - 1) \end{cases}$

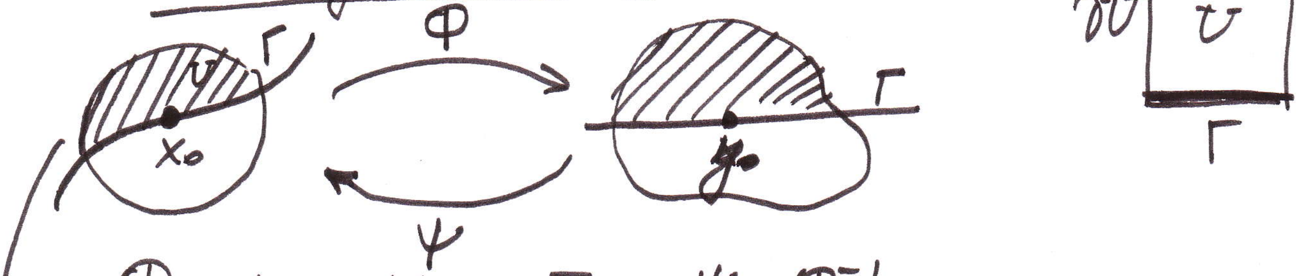
$$x^0 = \frac{4x_2 - x_1}{4}$$

$$e^s - 1 = \frac{x_1}{2x^0} = \frac{2x_1}{4x_2 - x_1}$$

$$e^s = \frac{4x_2 - x_1 + 2x_1}{4x_2 - x_1} = \frac{4x_2 + x_1}{4x_2 - x_1}$$

$$\Rightarrow u(x_1, x_2) = z(s) = \left(\frac{4x_2 - x_1}{4}\right)^2 \left(\frac{4x_2 + x_1}{4x_2 - x_1}\right)^2 = \frac{(4x_2 + x_1)^2}{16}$$

Boundary conditions:



Φ – straightens Γ , $\Psi = \Phi^{-1}$

$u(x) = v(\Phi(x)) \Rightarrow Du(x) = Dv(y) \cdot D\Phi(x)$

$$\begin{cases} F(Du, u, x) = 0 & \text{in } V \\ u = g & \text{on } \Gamma \end{cases} \qquad \begin{cases} G(Dv, v, y) = 0 & \text{in } V \\ v = h & \text{on } \Delta = \Phi(\Gamma) \end{cases}$$

Compatibility conditions: (for boundary data)

Assume Γ is flat near x^0 $\Rightarrow$ part of plane $\{x_n = 0\}$

1) (1) $\boxed{z^0 = g(x^0)}$ $\leftarrow$ because x should pass through x^0.

2) $u(x_1, \dots x_{n-1}, 0) = g(x_1, \dots x_{n-1})$ near x^0

$$p_i^0 = \begin{cases} u_{x_i}(x^0) = g_{x_i}(x^0), & i = 1, \dots n-1 \\ F(p^0, z^0, x^0) = 0 \end{cases}$$

$$\Rightarrow (2) \boxed{\begin{cases} p_i^0 = g_{x_i}(x^0) \\ F(p^0, z^0, x^0) = 0 \end{cases}}$$

n eqns in n unknowns $p^0 = (p_1^0, \dots p_n^0)$

(p^0, z^0, x^0) satisfying (1)-(2) is called admissible

z^0 – uniquely solved for from (1)

p^0 – can be non-unique, or might not exist.