Math 678.
Lecture 22.

Characteristics: method for solving 1st order nonlinear PDE.

$$\begin{cases} F(Du, u, x) = 0 & \text{in } U, \quad \Gamma \subseteq \partial U, \quad F, g\text{ - smooth} \\ u = g & \text{on } \Gamma \quad g: \Gamma \to \mathbb{R} \end{cases}$$

Goal: convert this PDE problem into a system of ODEs.

We want to construct a path (curve)

$$x(s) = (x_1(s) \dots x_n(s)), \quad s \in I \subseteq \mathbb{R}$$

s.t. we know solution along this path, starting with initial point $x^0 = x(0)$.

Define $z(s) := u(x(s))$ - soln along the path

$$p(s) := Du(x(s)), \quad p_i(s) = u_{x_i}(x(s))$$

Now we need to find suitable $x(s)$.

$$\begin{cases} (1) \quad \dot{p}_i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \cdot \dot{x}_j(s) \\ (2) \quad \sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s))\, u_{x_i x_j} + F_z(p(s), z(s), x(s))\, u_{x_i} \\ \qquad + F_{x_i}(p(s), z(s), x(s)) = 0. \end{cases}$$

Choose $\boxed{\dot{x}_j = F_{p_j}(p(s), z(s), x(s))} \Rightarrow$

$$\begin{cases} \dot{p}_i(s) = -F_{x_i}(p, z, x) - F_z(p, z, x)\, p_i \\ \text{Since } z(s) = u(x(s)) \Rightarrow \\ \dot{z}(s) = \sum_{j=1}^{n} \underbrace{u_{x_j}(x(s))}_{p_j} \cdot \underbrace{\dot{x}_j(s)}_{F_{p_j}} = \sum_{j=1}^{n} p_j(s) \cdot F_{p_j}(p, z, x) \\ \dot{x}(s) = F_{p_j}(p, z, x) \end{cases}$$

Now $\begin{cases} \dot{p}(s) = -D_x F(p,z,x) - D_z F(p,z,x)\, p(s) \\ \dot{z}(s) = D_p F(p,z,x) \cdot p(s) \\ \dot{x}(s) = D_p F(p,z,x) \end{cases}$

<u>Char. equations</u> in (p, z, x)

z - solution
x - characteristic
p - gradient

Moreover, $F(p(s), z(s), x(s)) = 0$

<u>Examples.</u> $F(p, z, x) = 0$

① Linear case

(a) $\begin{cases} u_t + u_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R} \end{cases}$ $\qquad X = (t, x)$

$F(p, z, X) = p_1 + p_2 = 0 \qquad D_p F = (1, 1) \qquad D_x F = D_z F = 0$

$p = (p_1, p_2)$

$X = (t, x)$

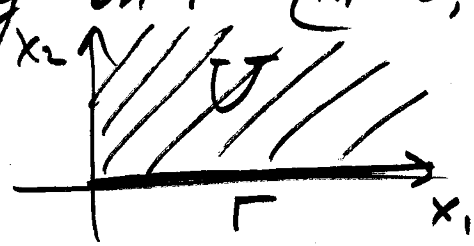
$\dot{X}(s) = D_p F \Rightarrow \begin{cases} \dot{t}(s) = 1 & t(0) = 0 \\ \dot{x}(s) = 1 & x(0) = x_0 \\ \dot{z}(s) = p_1 + p_2 = 0 & z(0) = u_0(x_0) \end{cases}$

$\Longleftarrow$

$x_0 = x - t \leftarrow \begin{cases} t = s \\ x = s + x_0 = x_0 + t & \leftarrow \text{straight line characteristic} \\ z = u_0(x_0) = \\ \quad = u_0(x - t) \end{cases}$

$\Rightarrow u(x,t) = z(s) = u_0(x-t) \leftarrow$ const along the characteristic not changing shape with velocity $= 1$.

(B) $\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U = \{x_1 > 0,\ x_2 > 0\} \\ u = g & \text{on } \Gamma = \{x_1 > 0,\ x_2 = 0\} \end{cases}$



$$F(p,z,x) = x_1 p_2 - x_2 p_1 - z = 0$$

$D_x F = (p_2, -p_1)$ $\qquad \dot{x}(s) = D_p F \Rightarrow \dot{x}_1 = -x_2$

$D_p F = (-x_2, x_1)$ $\qquad \dot{x}_2 = x_1$

$D_z F = -1$ $\qquad \dot{z}(s) = D_p F \cdot p \Rightarrow \dot{z} = -p_1 x_2 + x_1 p_2$

$\qquad \dot{z} = z$

$$\begin{cases} \dot{x}_1 = -x_2 & x_1(0) = x^0 \\ \dot{x}_2 = x_1 & x_2(0) = x^0 \\ \dot{z} = z & z(0) = z^0 \end{cases} \Rightarrow \begin{cases} x_1 = x^0 \cos s \\ x_2 = x^0 \sin s \end{cases}$$

$\Rightarrow z(s) = z^0 \cdot e^s = g(x^0) \cdot e^s$

$$x_1^2 + x_2^2 = x^{0\,2} \Rightarrow x^0 = \sqrt{x_1^2 + x_2^2}$$

$$\tan s = \frac{x_2}{x_1} \Rightarrow s = \arctan \frac{x_2}{x_1}$$

$$\Longrightarrow u(x_1(s), x_2(s)) = z(s) = g\left(\sqrt{x_1^2 + x_2^2}\right) \cdot e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

In general, if F is linear: $\boxed{b(x) Du + c(x) u = 0}$

$$F(p,z,x) = b(x) \cdot p + c(x) \cdot z$$

$D_p F = b(x)$, $\quad \boxed{\dot{x}(s) = b(x(s))}$

$$\dot{z}(s) = b(x(s)) \cdot p(s) \Rightarrow$$

$$\boxed{\dot{z}(s) = -c(x(s)) \cdot z(s)}$$

In this case, char. eqns are also linear.

1a Nonhomogeneous linear case.

$$\begin{cases} u_t - u_x = f(x,t) \text{ in } \mathbb{R} \times (0,\infty) \\ u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R} \end{cases}$$

$F(p,z,x) = p_1 - p_2 - f(x) = 0$, $\dot{x}(s) = D_p F = (1,-1)$

$\cancel{D_p F} = (1,-1)$

$D_p F \cdot p = p_1 - p_2$

$$\begin{cases} \dot{t} = 1 & t(0) = 0 \\ \dot{x} = -1 & x(0) = x_0 \\ \dot{z} = p_1 - p_2 = f & z(0) = u_0(x_0) \end{cases}$$

$$\Rightarrow \begin{cases} t = s \\ x = x_0 - s \quad \Rightarrow \quad x = x_0 - t \quad \text{- char. line} \\ \dot{z} = f(x,t) = f(x_0 - s, s) \end{cases}$$

$$z(s) = u_0(x_0) + \int_0^s f(x_0 - \tau, \tau)\, d\tau$$

$$\begin{Bmatrix} x_0 = x + t \\ s = t \end{Bmatrix} \Rightarrow u(x,t) = z(s) = u_0(x+t) + \int_0^t f(x_0 + t - \tau, \tau)\, d\tau$$

② Quasilinear.

$$F(Du, u, x) = b(x, u(x)) \cdot Du(x) + c(x, u(x)) = 0$$

$$\begin{cases} \dot{x} = b(x, z) \\ \dot{z} = b(x,z) \cdot p(s) = -c(x,z) \end{cases}$$

Ex. Burger's eqn.

$$\begin{cases} u_t + u \cdot u_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R} \end{cases}$$

$$F = p_1 + z \cdot p_2 = 0$$

$$b = (1, z) = D_p F, \; c = 0 \qquad \begin{cases} (\dot{x}_1 =)\, \dot{t} = 1 & t(0) = 0 \\ (\dot{x}_2 =)\, \dot{x} = z(s) & x(0) = x_0 \\ \dot{z} = 0 & z(0) = u_0(x_0) \end{cases}$$

$$\begin{cases} t = s \\ x = x_0 + u_0(x_0) \cdot s \\ z = u_0(x_0) \end{cases} \Rightarrow \quad \begin{array}{l} s = t \\ x_0 + u_0(x_0) \cdot t = x \\ \downarrow \text{ explicit soln} \\ x_0 = x_0(x,t) \text{ by IFT} \end{array}$$

$$\Rightarrow \left[u(x,t) = u_0(\underbrace{x_0(x,t)}) \right]$$