Math 678.

Lecture 21.

$n = 2k+1$

$$-\frac{1}{2r}\left(e^{-\lambda r^2}\right)_r = \lambda e^{-\lambda r^2}$$

$$\int_0^\infty u(x,s)e^{-\lambda s^2}ds = \frac{n\alpha(n)}{2}\left(\frac{\lambda}{\pi}\right)^{\frac{n-1}{2}}\int_0^\infty e^{-\lambda r^2} r^{n-1}G(x;r)\,dr$$

$$\lambda^{\frac{n-1}{2}}\int_0^\infty e^{-\lambda r^2} r^{n-1}G(x;r)\,dr = \int_0^\infty \lambda^k e^{-\lambda r^2} r^{2k}G(x;r)\,dr$$

$$= \frac{(-1)^k}{2^k}\int_0^\infty \left[\left(\frac{1}{r}\frac{d}{dr}\right)^k\left(e^{-\lambda r^2}\right)\right] r^{2k}G(x;r)\,dr =$$

$$= \frac{1}{2^k}\int_0^\infty r\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^k\left(r^{2k-1}G(x;r)\right)\right]e^{-\lambda r^2}dr$$

$$\int_0^\infty u(x,s)e^{-\lambda s^2}ds = \frac{n\alpha(n)}{\pi^{\frac{n-1}{2}}2^{k+1}}\int_0^\infty r\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^k\left(r^{2k-1}G\right)\right]e^{-\lambda r^2}dr$$

Let $\tau = r^2$ or $\tau = s^2$

$$\mathcal{L}\{u\} = \mathcal{L}\{\bar{g}\}, \quad \bar{g} = \frac{n\alpha(n)\ t}{\pi^{\frac{n-1}{2}}2^{k+1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^k\left(t^{2k-1}G\right)$$

$$\Rightarrow u(x,t) = \frac{n\alpha(n)\ t}{\pi^k\, 2^{k+1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^k\left(t^{2k-1}G(x,t)\right)$$

Coincides with previous result.

$$\frac{n\alpha(n)}{\pi^k 2^{k+1}} = \frac{1}{\gamma_n} \leftarrow (n-2)!!$$

Cole-Hopf transform:

Ex.
$$\begin{cases} u_t - a\Delta u + b|Du|^2 = 0, & \mathbb{R}^n\times(0,\infty) \\ u = g, & \mathbb{R}^n\times\{t=0\} \end{cases}$$

quasi-linear PDE

$w := \varphi(u)$ - smooth

We choose φ so that it satisfies linear equation : $w_t - a\Delta w = 0$.

$$w_t = \varphi'(u) u_t$$
$$\Delta w = \varphi'(u)\Delta u + \varphi''(u)|Du|^2$$

$$\Rightarrow w_t = \varphi'(u)u_t = \varphi'(u)(a\Delta u - b|Du|^2) =$$
$$= a\varphi'(u)\Delta u - b\varphi'(u)|Du|^2 =$$
$$= a(\varphi'(u)\Delta u + \varphi''(u)|Du|^2) -$$
$$- (a\varphi''(u) + b\varphi'(u))|Du|^2$$
$$= a\Delta w - (a\varphi'' + b\varphi')|Du|^2$$

If we choose φ s.t. $a\varphi'' + b\varphi' = 0 \Rightarrow$ $w_t = a\Delta w$

$\underline{\varphi(z) = e^{-\frac{b}{a}z}}$ $\quad z = \varphi'$

$az' + bz = 0$

$w = \varphi(u) = e^{-\frac{b}{a}u}$ $\quad z' = -\frac{b}{a}z \quad z = Ce^{-\frac{b}{a}u}$

$\varphi = -\frac{aC}{b}e^{-\frac{b}{a}u} + \tilde{C}$

$u = -\frac{a}{b}\ln w$, w - soln to linear heat eqn. $\begin{cases} w_t = a\Delta w \\ w = g \text{ on } \{t=0\} \end{cases}$

$$\Rightarrow u(x,t) = -\frac{a}{b}\ln\left(\frac{1}{(4\pi a t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at} - \frac{b}{a}g(y)}dy\right)$$

Holograph transform:

$$(\sigma^2(u) - (u^1)^2)u^1_{x_1} - u^1u^2(u^1_{x_2} + u^2_{x_1}) + (\sigma^2(u) - (u^2)^2)u^2_{x_2} = 0$$

u - unknown $= (u^1, u^2)$ $\quad (x_1, x_2)$ - indep. var.

σ - speed, known : $\mathbb{R}^2 \to \mathbb{R}$

Look at $x^1 = x^1(u_1, u_2)$
$\qquad\quad x^2 = x^2(u_1, u_2)$

IFT guarantees this is possible if

$$|J| = \left|\frac{\partial(u^1,u^2)}{\partial(x_1,x_2)}\right| = u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1} \neq 0$$

$$\begin{cases} u^2_{x_2} = |J|\, x^1_{u_1} \\ u^2_{x_1} = -|J|\, x^2_{u_1} \\ u^1_{x_2} = -|J|\, x^1_{u_2} \\ u^1_{x_1} = |J|\, x^2_{u_2} \end{cases} \qquad \begin{aligned} x^2_{u_1} &= -\frac{1}{|J|}u^2_{x_1} \\ x^1_{u_2} &= -\frac{1}{|J|}u^1_{x_2} \end{aligned}$$

$$\begin{cases} \cancel{|J|}\left(\sigma^2(u) - u_1^2\right)x^2_{u_2} + u_1u_2\left(x^1_{u_2} + x^2_{u_1}\right) + \left(\sigma^2(u) - u_2^2\right)x^1_{u_1} = 0 \\ x^1_{u_2} - x^2_{u_1} = 0 \end{cases}$$

linear system

Use $z = z(u)$ s.t. $\begin{cases} x^1 = z_{u_1} \\ x^2 = z_{u_2} \end{cases}$ then

$$\left(\sigma^2(u) - u_1^2\right)z_{u_2u_2} + 2u_1u_2\, z_{u_1u_2} + \left(\sigma^2(u) - u_2^2\right)z_{u_1u_1} = 0$$

2nd order linear PDE in z

Burger's equation with viscosity:

$$\begin{cases} u_t - a u_{xx} + u\cdot u_x = 0 & \mathbb{R}\times(0,\infty) \\ u = g & \mathbb{R}\times\{t=0\} \end{cases}$$

$$w(x,t) = \int_{-\infty}^{x} u(y,t)\,dy \qquad w_x = u, \quad w_{xx} = u_x$$

$$h(x) = \int_{-\infty}^{x} g(y)\,dy \qquad h_x = g$$

$$(w_x^2)_x = 2w_x\cdot w_{xx} = 2u\cdot u_x$$

$$\Rightarrow \boxed{w_t - a w_{xx} + \frac{1}{2}w_x^2 = 0, \quad w(\cdot,0) = h} \qquad n=1, \ b=\frac{1}{2}$$

$\tilde{w} = \varphi(w)$ Cole-Hopf transform of w

$$w(x,t) = -2a\ln\left(\frac{1}{(4\pi a t)^{1/2}}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}}\,dy\right)$$

$w_x = u$

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \frac{x-y}{t} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}}\, dy}{\displaystyle\int_{-\infty}^{+\infty} e^{-\frac{|x-y|^2}{4at} - \frac{h(y)}{2a}}\, dy}$$