Math 678.
Lecture 20.

$$\hat{u} = \mathcal{F}(u)$$

$$\begin{cases} u_{tt} - \Delta u = 0 & \mathbb{R}^n \times (0,\infty) \\ u = g & \\ u_t = h & t=0 \end{cases}$$

$$\begin{cases} \hat{u}_{tt} + |y|^2 \hat{u} = 0 \\ \hat{u} = \hat{g} \\ \hat{u}_t = \hat{h} \end{cases} \Longrightarrow \hat{u} = \hat{g}\cos t|y| + \frac{\hat{h}}{|y|}\sin t|y|$$

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{g}\cos t|y| + \frac{\hat{h}}{|y|}\sin t|y|\right)$$

$$h = 0 \Rightarrow u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\hat{g}(y)}{2}\left(e^{i(x\cdot y + t|y|)} + e^{i(x\cdot y - t|y|)}\right) dy$$

Energy method for wave eqn: $E(t) := \frac{1}{2}\int_{\mathbb{R}^n} (u_t^2 + |Du|^2)\,dx$

Recall $E(t) = E(0) = \frac{1}{2}\int_{\mathbb{R}^n} (h^2 + |Dg|^2)\,dx = \text{const}$

<u>To show:</u> $\lim_{t\to\infty} \int_{\mathbb{R}^n} |Du|^2 dx = \lim_{t\to\infty} \int_{\mathbb{R}^n} u_t^2\,dx = E(0)$

$$\hat{u} = \hat{g}\cos t|y| + \frac{\hat{h}}{|y|}\sin t|y|$$

$$\int_{\mathbb{R}^n} |Du|^2 dx = \int_{\mathbb{R}^n} |y|^2 |\hat{u}|^2 dy =$$

$$= \int_{\mathbb{R}^n} \Big(\overbrace{|y|^2|\hat{g}|^2\cos^2 t|y|}^{\frac{1}{2}|y|^2|\hat{g}|^2} + \overbrace{|\hat{h}|^2 \sin^2 t|y|}^{\frac{1}{2}|\hat{h}|^2}\Big)\,dy$$

$$+ \int_{\mathbb{R}^n} \underbrace{\cos t|y| \sin t|y| \cdot |y| (\hat{h}\bar{\hat{g}} + \bar{\hat{h}}\hat{g})\,dy}_{\downarrow \text{ as } t\to\infty \atop 0}$$

$$\cos^2 t|y| = \frac{1+\cos 2t|y|}{2}$$

$$\int_{\mathbb{R}^n} f \cos t|y| \cdot \sin t|y| \, dy = \frac{1}{2}\int_{\mathbb{R}^n} \sin 2t|y| \cdot f =$$

$$= \frac{1}{2}\int_0^\infty \sin(2tr) \cdot \int_{\partial B(0,r)} f \, dS \, dr = -\frac{1}{4r}\int_0^\infty \frac{d}{dr}(\cos 2tr) \int_{\partial B(0,r)} f \, dS \, dr$$

$$= \frac{1}{4t}\int_0^\infty \cos 2tr \cdot \frac{d}{dr}\Big(\int_{\partial B(0,r)} f \, dS\Big) dr \xrightarrow[t\to\infty]{} 0$$

$$\Rightarrow \lim_{t\to\infty} \int_{\mathbb{R}^n} |Du|^2 dx = \frac{1}{2}\Big(\int_{\mathbb{R}^n} |y|^2 |\hat{g}|^2 dy + \int_{\mathbb{R}^n} |\hat{h}|^2 dy\Big) =$$

$$= \frac{1}{2}\Big(\int_{\mathbb{R}^n} |Dg|^2 dy + \int_{\mathbb{R}^n} |h|^2 dy\Big) = E(0)$$

Radon transform:

S^{n-1} – unit sphere in $\mathbb{R}^n$

$\Pi(s,\omega) := \{y \in \mathbb{R}^n \mid y \cdot \omega = s\}$ – plane in $\mathbb{R}^n$

(ω: unit normal; s: distance to origin)

Def. $(\mathcal{R}u)(s,\omega) := \int_{\Pi(s,\omega)} u \, dS$, $\quad s \in \mathbb{R}$, $\omega \in S^{n-1}$

$(\mathcal{R}u)(s,\omega) = \tilde{u}(s,\omega)$

Properties:

1) $\tilde{u}(-s,-\omega) = \tilde{u}(s,\omega)$

2) $\mathcal{R}(D^\alpha u) = \omega^\alpha \frac{\partial^{|\alpha|}}{\partial s^{|\alpha|}} \tilde{u} = \omega^\alpha \frac{\partial^{|\alpha|}}{\partial s^{|\alpha|}} \mathcal{R}(u)$

3) $\mathcal{R}(\Delta u) = \frac{\partial^2}{\partial s^2}\tilde{u} = \frac{\partial^2}{\partial s^2}\mathcal{R}(u)$

4) If $u \equiv 0$ in $\mathbb{R}^n \setminus B(0,R) \Rightarrow$

$\tilde{u}(s,\omega) = 0$ for $|s| \geq R$

Thm. (Inverting Radon transform).

1) $$u(x) = \frac{1}{2(2\pi)^n} \int_{\mathbb{R}} \int_{S^{n-1}} \tilde{u}(r,\omega) r^{n-1} e^{ir\omega\cdot x} \, dS \, dr$$

2) $n = 2k+1$ odd $\Rightarrow$

$$u(x) = \int_{S^{n-1}} r(x\cdot\omega, \omega) \, dS$$

$$r(s,\omega) := \frac{(-1)^k}{2(2\pi)^{2k}} \frac{\partial^{2k}}{\partial s^{2k}} \tilde{u}(s,\omega)$$

It follows that if n - odd,

$\tilde{u} = 0$ for $|s| \le R$ then $u \equiv 0$ in $B(0,R)$

Laplace transform:

$$\mathcal{L}u(s) := \int_0^\infty e^{-st} u(t) \, dt \quad , s \ge 0$$

Derivation of wave eqn. from heat eqn:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \left.\begin{aligned} u &= g \\ u_t &= 0 \end{aligned}\right\} & t = 0 \end{cases}$$

n - odd

g - smooth compact support

Extend it to $t \in \mathbb{R}$ by $u(x,t) = u(x,-t)$, $t < 0$

$\Rightarrow \quad u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$

$$v(x,t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} u(x,s) \, ds \quad , \quad t > 0$$

$$\xrightarrow{t \to 0} g$$

$$\Delta v = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} \Delta u(x,s) \, ds =$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} u_{ss}(x,s) \, ds =$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \frac{s}{2t} e^{-s^2/4t} u_s(x,s) \, ds$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \left(\frac{s^2}{4t^2} - \frac{1}{2t}\right) e^{-s^2/4t}\, u(x,s)\, ds = v_t(x,t)$$

$$\Rightarrow \begin{cases} v_t - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad \text{heat eqn}$$

$$\Rightarrow v = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad \text{from fundam. soln for heat}$$

Reconcile

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t}\, u(x,s)\, ds \quad =$$

$$\lambda = \frac{1}{4t} \qquad u(x,-s) = u(x,s)$$

$$\sqrt{\frac{\lambda}{\pi}} \int_0^{\infty} u(x,s)\, e^{-\lambda s^2}\, ds = \left(\sqrt{\frac{\lambda}{\pi}}\right)^n \int_{\mathbb{R}^n} e^{-\lambda |x-y|^2} g(y)\, dy =$$

$$= \frac{n\,\alpha(n)}{2} \left(\frac{\lambda}{\pi}\right)^n \int_0^{\infty} e^{-\lambda r^2} r^{n-1}\, G(x,r)\, dr$$

$$G(x,r) = ⨍_{\partial B(x,r)} g(y)\, dS(y)$$