Math 678.

Lecture 2.

Transport eqn: IVP $\begin{cases} u_t + b\cdot Du = 0 \\ u(x,0) = g(x) \end{cases}$ (homogeneous)

$\Rightarrow \boxed{u(x,t) = g(x - tb)}$

Nonhom. transport eqn: IVP $\begin{cases} u_t + b\cdot Du = f \\ u(x,0) = g(x) \end{cases}$

$\Rightarrow \boxed{u(x,t) = g(x - tb) + \int_0^t f(x+(s-t)b, s)\, ds}$

§2.2. Laplace's eqn.

$\Delta u = 0$ Laplace $\qquad u : \bar{U} \to \mathbb{R}$, $u = u(x)$

$-\Delta u = f$ Poisson $\qquad U \subset \mathbb{R}^n$ - open set

Laplacian: $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$

Def. If $\Delta u = 0$, $u \in C^2$ it is called a harmonic function.

Physical intuition behind Laplace eqn:

u - density in equilibrium

V - smooth subregion in U

F - flux density, ν - outward normal

$\Rightarrow \int_{\partial V} F\cdot\nu\, dS = 0$ ← total flux trough surface

$\|$ by Gauss-Green Thm (Divergence Thm)

$\int_V \operatorname{div} F\, dx \implies \operatorname{div} F = 0$ since V is arbitrary.

Take $F \propto Du$

Since flux is directed toward lower concentration $\Rightarrow F = -aDu$, $a>0$

$\Rightarrow \quad \operatorname{div}(Du) = \boxed{\Delta u = 0}$

Finding fundamental solution of $\Delta u = 0$:

Observation: $\Delta u = 0$ is rotationally invariant

i.e. $\Delta u(Rx) = 0$ with R - orthogonal matrix.

We will try to find a radial solution:

$u(x) = v(r)$, $r = \sqrt{x_1^2 + .. + x_n^2}$

Let us find v that satisfies the $\Delta v = 0$ eqn.

$$\frac{\partial r}{\partial x_i} = \frac{2x_i}{2\sqrt{x_1^2+..+x_n^2}} = \frac{x_i}{r}$$

$$u_{x_i} = v'(r)\cdot\frac{\partial r}{\partial x_i} = v'(r)\cdot\frac{x_i}{r} \qquad \left(\frac{x_i}{r}\right)' = \frac{1\cdot r - x_i\cdot\frac{x_i}{r}}{r^2}$$

$$u_{x_i x_i} = v''(r)\cdot\left(\frac{x_i}{r}\right)^2 + v'(r)\cdot\left(\frac{x_i}{r}\right)' \qquad = \frac{1}{r} - \frac{x_i^2}{r^3}$$

$$= v''(r)\cdot\frac{x_i^2}{r^2} + v'(r)\cdot\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), \quad i = 1, .. n$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = v''(r) + n\cdot\frac{v'}{r} - \frac{v'}{r}$$

For $\Delta u = 0$ we need $v'' + \frac{n-1}{r}\cdot v' = 0$

$\Rightarrow \frac{v''}{v'} = + \frac{1-n}{r}$ separable

$$(\log v')' = \frac{1-n}{r} \Rightarrow \log v' = (1-n)\log r + C$$

$$v' = ae^{(1-n)\log r} = \frac{a}{r^{n-1}}$$

So if $r>0 \Rightarrow v(r) = \begin{cases} \tilde{a}\log r + c, & n=2 \\ \frac{\tilde{a}}{r^{n-2}} + c, & n \geq 3 \end{cases}$

Def. $\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & , \quad n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} , & n \geq 3 \end{cases}$

$\alpha(n) =$ ~~area~~ volume of an n-sphere (in $\mathbb{R}^n$) of radius 1.

$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, [Volume of a ball $B(0,r)$ in $\mathbb{R}^n$ is : $\alpha(n)\cdot r^n$]

Surface area of $B(0,r)$ in $\mathbb{R}^n$: $n(\alpha(n))\cdot r^{n-1}$

Properties of $\Phi(x)$: 1) $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$

2) $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$, $x \neq 0$

Poisson eqn case: $\Delta u = f$

Solution is expected in the form:

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \quad (*)$$

Plug in argument does not work since

$$\Delta u = \int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\, dy$$

So we need a more careful argument.

Thm. Define $u(x)$ by $(*)$. Then

1) $u \in C^2(\mathbb{R}^n)$

2) $-\Delta u = f$ in $\mathbb{R}^n$

provided $f \in C_c^2(\mathbb{R}^n)$ - 2-times cont. diff. with compact support.

1) $\dfrac{u(x+he_i)-u(x)}{h} = ?$

First notice that $\displaystyle\int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \Phi(\tilde{y}) f(x-\tilde{y})\,d\tilde{y}$

$\tilde{y} = x-y \Rightarrow y = x-\tilde{y}$

$d\tilde{y} = -dy$

$$\int_{-\infty}^{+\infty} \Phi(x-y) f(y)\,dy = -\int_{+\infty}^{-\infty} \Phi(\tilde{y}) f(x-\tilde{y})\,d\tilde{y}$$

$$\Rightarrow u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,dy$$

$$\Rightarrow \frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \cdot \underbrace{\frac{f(x+he_i-y)-f(x-y)}{h}}_{\downarrow\ \frac{\partial f}{\partial x_i}}\,dy$$

$e_i = (0, \ldots \underset{i}{1}, \ldots 0)$

$$\Rightarrow \frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x-y)\,dy$$

Same way for $\displaystyle\frac{\partial^2 u}{\partial x_i^2}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\,dy$ $\quad i, j = 1, \ldots n$

$\Rightarrow u \in C^2(\mathbb{R}^n)$.

2) Now let's show $-\Delta u = f$ in $\mathbb{R}^n$.

Problem: Φ blows up at 0.

$$\Delta u(x) = \underbrace{\int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{J_\varepsilon}$$

$$|I_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y) \cdot \Delta_x f(x-y)\,dy \right| \le C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \left| \int_{B(0,\varepsilon)} \Phi(y)\,dy \right|$$

Recall $\Phi(x) = \begin{cases} \frac{-1}{2\pi} \log|x| \,, & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} \,, & n \geq 3 \end{cases}$

$n=2$ $\quad \left| \int_{B(0,\varepsilon)} \Phi(y)\,dy \right| \leq C \log\varepsilon \cdot \varepsilon^2$

$$\int_{B(0,\varepsilon)} dy = \pi\varepsilon^2$$

$$\left| \int_{B(0,\varepsilon)} \Phi(y)\,dy \right| = \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y)\, dS(r) \cdot dr = \int_0^\varepsilon \frac{1}{2\pi} \int_{\partial B(0,r)} \log r \, dS(r)\, dr$$

$$= \int_0^\varepsilon \frac{1}{2\pi} \cdot \log r \cdot \Big(\underbrace{\int_{\partial B(0,r)} dS(r)}_{2\pi r} \Big)\, dr = \int_0^\varepsilon r \cdot \log r \, dr$$

$$= \frac{\varepsilon^2 \log\varepsilon}{2} - \frac{\varepsilon^2}{4}$$