Solitons:

\[ u_t + 6u \cdot u_x + u_{xxx} = 0 \quad \text{KdV eqn} \]

\[ u(x,t) = v(x-6t) \quad \text{\(\sigma\)-velocity of traveling wave} \]

\[-6v' + 6v \cdot v' + v'' = 0 \]

\[-6v + 3v^2 + v'' = A = \text{const} \quad (1) \]

\[-6v \cdot v' + 3v^2 v' + v'' v' = A v' \]

\[ v'' v' = A v' + 5v \cdot v' - 3v^2 v' \]

\[ \left( \frac{(v')^2}{2} \right)' = (Av)' + \left( \frac{3v^2}{2} \right)' - (v^3)' \]

\[ \frac{(v')^2}{2} = Av + \frac{3v^2}{2} - v^3 + B \quad (2) \]

Take \( s \to \pm \infty \Rightarrow \text{we are interested in} \ v \ \text{s.t.} \]

\[ v, v', v'' \to 0 \]

\[ A = 0 \quad \text{from (1)} \]

\[ B = 0 \quad \text{from (2)} \]

\[ \Rightarrow \left( \frac{v'}{2} \right)^2 = v^2 \left( \frac{v}{2} - v \right) \]

\[ (v')^2 = v^2 \left( \sigma - 2v \right) \]

\[ v' = \pm v \sqrt{\sigma - 2v} \quad \text{pick} \ \theta \]

\[ \int \frac{dv}{v \sqrt{\sigma - 2v}} = - \int ds \]

\[ v(s) = \frac{\sigma}{2} \text{sech}^2 \left( \frac{\sqrt{\sigma}}{2} (s - c) \right) \]

\[ u(x,t) = \frac{\sigma}{2} \text{sech}^2 \left( \frac{\sqrt{\sigma}}{2} (x - \sigma t - c) \right) - \text{soliton wave} \]

\[ \text{Soln to KdV eqn.} \]
Transform methods.

1. Fourier transform.

**Def.** \( u \in L^1(\mathbb{R}^n) \)

*direct* \( \hat{u}(y) = \mathcal{F}(u) : = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot y} u(x) \, dx \), \( y \in \mathbb{R}^n \)

*inverse* \( \check{u}(y) = \mathcal{F}^{-1}(\hat{u}) : = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} \hat{u}(x) \, dx \), \( y \in \mathbb{R}^n \)

**Plancherel's Thm.:**

\( u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow \hat{u}, \check{u} \in L^2(\mathbb{R}^n) \) and

\[ \| \hat{u} \|_{L^2(\mathbb{R}^n)} = \| \check{u} \|_{L^2(\mathbb{R}^n)} = \| u \|_{L^2(\mathbb{R}^n)} \]

Consider \( \{ u_k \}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and

\( u_k \to u \) in \( L^2(\mathbb{R}^n) \)

\[ \| \hat{u}_k - \hat{u}_j \|_{L^2(\mathbb{R}^n)} = \| (u_k - u_j) \|_{L^2(\mathbb{R}^n)} = \| u_k - u_j \|_{L^2(\mathbb{R}^n)} \]

\( \Rightarrow \{ \hat{u}_k \}_{k=1}^{\infty} - \text{Cauchy sequence} \Rightarrow \text{converges to} \ \hat{u} \).

\( \hat{u} \) - is called a Fourier transform of \( u \)

\( u \) does not depend on the choice of \( k \in \mathbb{Z}^n \).

**Properties.**

1) \( \partial^\alpha \hat{u} = (iy)^\alpha \hat{u} \), \( \alpha \) - multi-index

2) \( \int_{\mathbb{R}^n} u \cdot \overline{v} \, dx = \int_{\mathbb{R}^n} \hat{u} \cdot \overline{\hat{v}} \, dy \)

3) \( u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \Rightarrow (\hat{u} \ast \hat{v}) = (2\pi)^{n/2} \overline{\hat{u}} \cdot \hat{v} \)

4) \( u = (\hat{u})^\vee = \mathcal{F}^{-1}(\mathcal{F}(u)) \)
Example

1) Heat eqn IVP: \[ \begin{align*}
\hat{u} - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
\hat{u} &= \hat{g} \quad \text{on } \mathbb{R}^n \times \{ t = 0 \}
\end{align*} \]

\[ \hat{u} = \hat{F}(u) = \hat{u}(y) \quad y \in \mathbb{R}^n \]

\[ \begin{align*}
\hat{F}(u_t - \Delta u) &= 0 \\
\hat{u}_t + |y|^2 \hat{u} &= 0 \quad \text{as } t > 0 \\
\hat{u} &= \hat{g} \quad \text{as } t = 0
\end{align*} \]

\[ u(x, t) = \mathcal{F}^{-1} \left( e^{-t |y|^2} \hat{g} \right) = \mathcal{F}^{-1} \left( \hat{F} \cdot \hat{g} \right) = \frac{1}{(2\pi)^{n/2}} (\hat{F} \ast \hat{g}) \]

\[ \mathcal{F}(F \ast g) = (2\pi)^{n/2} \mathcal{F} \cdot \hat{g} \]

\[ \frac{1}{(2\pi)^{n/2}} \mathcal{F} \ast \hat{g} = \mathcal{F}^{-1} \left( \frac{1}{(2\pi)^{n/2}} \mathcal{F} \cdot \hat{g} \right) \]

\[ F = \mathcal{F}^{-1} \left( e^{-t |y|^2} \right) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t |y|^2} e^{ix \cdot y} dy = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t |y|^2 + ix \cdot y} dy \]

\[ \int_{-\infty}^{\infty} e^{-ax - bx^2} dx = \int_{-\infty}^{\infty} \left( e^{\frac{-a^2}{4B}} \right) e^{-u^2} dx = e^{\frac{-a^2}{4B}} \sqrt{\pi} \]

\[ a \cdot x - b x^2 = \frac{-a^2}{4B} - (\sqrt{B} x - \frac{a}{2B})^2 \]

\[ u = \sqrt{B} x - \frac{a}{2B} i \quad b x^2 - a \cdot x^2 + \frac{a^2}{4B} \quad B = +t \quad a = x_i \]

\[ \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} e^{-t y_j^2 + i x_j y_j} dy_j \right) = \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^{n} \left( e^{\frac{-x_j^2}{4t}} \sqrt{\pi} \right) \]

\[ = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{4t} \sum_{j=1}^{n} x_j^2} \Rightarrow u(x, t) = \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{(x - y)^2}{4t}} g(y) dy \]