Math 678.
Lecture 16.

$n = 2k+1$

$\frac{n-3}{2} = k-1$

$$\begin{cases} u_{tt} = \Delta u, & x \in \mathbb{R}^n \times [0,\infty) \\ u = g & \\ u_t = h, & t = 0 \end{cases}$$

U, G, H - spherical means

$\tilde{U} = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}} \left(r^{n-2} U\right)$, same for $\tilde{G}$, $\tilde{H}$.

1) To show: $\tilde{U}$ satisfies 1d wave eqn.
2) To show: $\lim_{r\to 0} \frac{\tilde{U}}{\gamma_n \cdot r} = u(x,t)$

This gives: $u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} \fint_{\partial B(x,t)} g \, dS\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h \, dS\right)\right]$

Follows from:

$$\frac{d^2}{dr^2}\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\varphi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\varphi}{dr}\right) \quad (1)$$

$$\frac{1}{r^{2k}} \cdot \frac{\partial}{\partial r}\left(r^{2k}\frac{\partial U}{\partial r}\right) = U_{rr} + \frac{2k}{r} U_r \quad (2)$$

$$\tilde{U}_{rr} = \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1} U\right) \underset{(1)}{=} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial U}{\partial r}\right) =$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}\left[r^{-2k}\frac{\partial}{\partial r}\left(r^{2k}\frac{\partial U}{\partial r}\right)\right] =$$

$$\underset{(2)}{=} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}\Big[\underbrace{U_{rr} + \frac{2k}{r}U_r}_{U_{tt} \text{ by Darboux}}\Big] = \tilde{U}_{tt}$$

To justify (2) :

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\varphi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\varphi}{dr^j}(r) = \beta_0^k r\cdot\varphi + \dots$$

$$\beta_0^k = (2k-1)!!$$

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left(r^{2k-1}\varphi(r)\right) = (2k-1)\cdot r^{2k-3}\varphi + r^{2k-2}\varphi'(r)$$

$$\left(\frac{1}{r}\frac{d}{dr}\right)^2\left(r^{2k-1}\varphi(r)\right) = (2k-1)(2k-3)r^{2k-5}\varphi + \underline{A}\cdot r^{2k-4}\varphi'(r) + B\cdot r^{2k-3}\varphi''(r)$$

By induction:

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{\ell}\left(r^{2k-1}\varphi(r)\right) = \left[(2k-1)\cdots(2k-2\ell+1)\right]r^{2k-2\ell-1}\varphi + \sum_{j=1}^{\ell}\beta_j^{\ell k} r^{2k-2\ell+j-1}\varphi^{(j)}(r)$$

$\beta_0^{\ell k} \Rightarrow$

$$\gamma_n = \beta_0^k = (2k-1)!! \qquad n = 2k+1 \quad k = \frac{n-1}{2}$$

$$\Rightarrow \lim_{r\to 0}\frac{\tilde{U}}{\gamma_n r} = \lim_{r\to 0}\left[U + (\cdot)\cdot r\frac{\partial U}{\partial r} + (\cdot)r^2\frac{\partial^2 U}{\partial r^2} + \dots + (\cdot)r^k\frac{\partial^k U}{\partial r^k}\right]$$

$$= \lim_{r\to 0} U(r) = u(x,t).$$

$\boxed{\text{Case } n = 2k}$

$u \in C^m$ – solves $\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g \\ u_t = h & \mathbb{R}^n\times\{t=0\}\end{cases}$

$m = \frac{n+2}{2}$

Derivation of the representation formula:

$$\begin{cases} \bar{u}(x_1,\dots x_{n+1},t) = u(x_1,\dots x_n,t) \\ \bar{u} = \bar{g} \\ \bar{u}_t = \bar{h} \end{cases} \text{ on } \mathbb{R}^{n+1}\times\{t=0\}$$

sol. of wave IVP in $\mathbb{R}^n$

$n+1$ - odd $\Rightarrow$ we know formula for $\bar{u}$
Then use descent method.

Fix $x \in \mathbb{R}^n$, $t>0$

$$\bar{x} = (x_1, \dots x_n, 0) \in \mathbb{R}^{n+1}$$

$$u(x,t) = \frac{1}{\gamma_{n+1}}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} g \, d\bar{S}\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1} \fint_{\partial \bar{B}(\bar{x},t)} h \, d\bar{S}\right)\right]$$

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g} \, d\bar{S} = \frac{1}{(n+1)\alpha(n+1) t^n} \int_{\partial \bar{B}(\bar{x},t)} \bar{g} \, d\bar{S}$$

$$\partial \bar{B}(\bar{x},t) = \partial^+ \bar{B}(\bar{x},t) \cup \partial^- \bar{B}(\bar{x},t)$$

$$\partial^+ \bar{B}(\bar{x},t) = \partial \bar{B}(\bar{x},t) \cap \{y_{n+1} \geq 0\}$$

$$\partial^- \bar{B}(\bar{x},t) = \partial \bar{B}(\bar{x},t) \cap \{y_{n+1} < 0\}$$

on upper hemisphere $\partial^+ \bar{B}(\bar{x},t)$ we have $\gamma(y) := \sqrt{t^2 - |y-x|^2}$ (same on $\partial^-(\bar{B}(\bar{x},t))$ but with a minus sign).

$$\int_{\partial \bar{B}(\bar{x},t)} \bar{g} \, d\bar{S} = \int_{B(x,t)} g(y) \cdot \sqrt{1 + |D_y \gamma|^2} \, dy = \int_{B(x,t)} g(y) \frac{t}{\sqrt{t^2 - |y-x|^2}} \, dy$$

$$\left|\frac{\partial \gamma}{\partial y}\right| = \frac{+\cancel{2}|y-x|}{\cancel{2}\sqrt{t^2 - |y-x|^2}}$$

$$1 + |D_y \gamma|^2 = \frac{t^2 - \cancel{|y-x|^2} + \cancel{|y-x|^2}}{t^2 - |y-x|^2}$$

$$\Rightarrow \fint_{\partial \bar{B}(\bar{x},t)} \bar{g} \, d\bar{S} = \frac{2}{(n+1)\alpha(n+1) t^{n-1}} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} \, dy =$$

$$|B(x,t)| = \cancel{n}\alpha(n) \cdot t^n$$

$$= \frac{2\alpha(n) t}{(n+1)\alpha(n+1)} \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} \, dy$$

$$u(x,t) = \frac{1}{\gamma_{n+1}} \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left[\fint_{B(x,t)} \frac{g(y)\cdot t^n}{\sqrt{t^2-|y-x|^2}}\,dy \right] \right.$$

$$= \frac{1}{\gamma_n}$$

$$\left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy \right) \right]$$

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \qquad \gamma_{n+1} = (n-1)!!$$

$$\gamma_n = 1\cdot 2\cdot 4 \cdot \ldots \cdot (n-2)\cdot n = n!!$$