Math 678.
Lecture 14.

$n=1$:
$$\begin{cases} u_{tt} - u_{xx} = 0 & \mathbb{R}\times(0,\infty) \\ \left.\begin{array}{l} u = g \\ u_t = h \end{array}\right\} & \mathbb{R}\times\{t=0\} \end{cases}$$

d'Alembert formula for solution:
$$u(x,t) = \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

Reflection method: $\mathbb{R}_+ = \{x>0\}$
$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{on } \mathbb{R}_+\times(0,\infty) \\ u = g,\ u_t = h & \text{on } \{t=0\}, \quad g(0)=h(0)=0 \\ u = 0 & \text{when } \{x=0\}\times(0,\infty) \end{cases}$$

Odd reflection:
$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x\geq 0,\ t\geq 0 \\ -u(-x,t), & x\leq 0,\ t\geq 0 \end{cases}$$
$$\tilde{g} = \begin{cases} g(x), & x\geq 0 \\ -g(-x), & x\leq 0 \end{cases}, \quad \tilde{h} = \begin{cases} h(x), & x\geq 0 \\ -h(-x), & x\leq 0 \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{u}_{tt} = \tilde{u}_{xx} \\ \left.\begin{array}{l}\tilde{u} = \tilde{g} \\ \tilde{u}_t = \tilde{h}\end{array}\right. \text{ at } \mathbb{R}\times\{t=0\} \end{cases}$$

By d'Alembert formula:
$$\tilde{u}(x,t) = \frac{1}{2}\left(\tilde{g}(x+t) + \tilde{g}(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy$$

When $x\geq 0,\ t\geq 0$
$$u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) - g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy, & x\geq t\geq 0. \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,dy, & 0\leq x\leq t \end{cases}$$

Spherical means.

In $\mathbb{R}^n$:
$$(*)\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = g \\ u_t = h & , \{t=0\} \end{cases} \qquad \begin{matrix} n \geq 2 \\ m \geq 2 \end{matrix}$$

$u \in C^m(\mathbb{R}^n \times [0,\infty))$

Idea: take averages over spheres, prove that they satisfy a certain ODE then use d'Alembert formula.

Notation: $U(x;r,t) = \displaystyle\fint_{\partial B(x,r)} u(y,t)\, dS(y)$

$G(x;r) = \displaystyle\fint_{\partial B(x,r)} g(y)\, dS(y)$, $H(x;r) = \displaystyle\fint_{\partial B(x,r)} h(y)\, dS(y)$

Lemma.

For fixed $x \in \mathbb{R}^n$, u - solution to $(*)$ $\Rightarrow$

$U(x;r,t) \in C^m(\overline{\mathbb{R}_+} \times [0,\infty))$ and satisfies.

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ U = G \\ U_t = H & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$$

Euler-Poisson-Darboux IVP.

Proof: $U(x;r,t) = \displaystyle\fint_{\partial B(x,r)} u(y,t)\, dS(y)$

$U_r(x;r,t) = \frac{r}{n} \displaystyle\fint_{B(x,r)} \Delta u(y,t)\, dy$ (same as in MVT.)

$\lim\limits_{r\to 0} U_r = 0$

$U_{rr}(x;r,t) = \displaystyle\fint_{\partial B(x,r)} \Delta u\, dS + \left(\frac{1}{n} - 1\right) \fint_{B(x,r)} \Delta u\, dy$

$\lim\limits_{r\to 0} U_{rr}(x;r,t) = \frac{1}{n} \Delta u(x,t)$

From here $U \in C^m(\overline{\mathbb{R}_+} \times [0,\infty))$

$$V_r = \frac{r}{n} \fint_{B(x,r)} \Delta u(y,t)\,dy \underset{\text{by } \circledast}{=} \frac{r}{n} \fint_{B(x,r)} u_{tt}\,dy = \frac{r}{n\alpha(n) r^n} \int_{B(x,r)} u_{tt}\,dy$$

$$= \frac{1}{n\alpha(n)\cdot r^{n-1}} \int_{B(x,r)} u_{tt}\,dy$$

$$r^{n-1} V_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} u_{tt}\,dy$$

$$(r^{n-1} V_r)_r = \frac{1}{n\alpha(n)} \frac{\partial}{\partial r}\left(\int_{B(x,r)} u_{tt}\,dy\right) = \frac{1}{n\alpha(n)} \frac{\partial}{\partial r}\left(\int_0^r \int_{\partial B(x,s)} u_{tt}\,dS\,ds\right)$$

$$\boxed{(r^{n-1} V_r)_r = (n-1) r^{n-2} V_r + r^{n-1} V_{rr}}$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\,dS =$$

$$= \frac{r^{n-1}}{n\alpha(n) r^{n-1}} \cdot \int_{\partial B(x,r)} u_{tt}\,dS =$$

$$= r^{n-1} \cdot \fint_{\partial B(x,r)} u_{tt}\,dS = \boxed{r^{n-1} V_{tt}}$$

$$\to (n-1) r^{n-2} V_r + r^{n-1} V_{rr} = r^{n-1} \cdot V_{tt}$$

$$\begin{cases} V_{tt} - V_{rr} - \frac{n-1}{r} \cdot V_r = 0 & \text{on } \mathbb{R}_+ \times (0,\infty) \\ \left.\begin{matrix} V = G \\ V_t = H \end{matrix}\right\} & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$$

Deriving solution in $\mathbb{R}^n$.

$\boxed{n=3}$ $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves $\circledast$.

Notice $V_{rr} + \frac{2}{r} V_r = \frac{1}{r^2}(r^2 V_r)_r = \frac{1}{r}(rV)_{rr}$

By EPD equation: $V_{rr} + \frac{2}{r} V_r = V_{tt}$

$$\Rightarrow V_{tt} = \frac{1}{r}(rV)_{rr}$$

$$r V_{tt} = (rV)_{rr}$$

Denote $\tilde{V} = rV \Rightarrow$ $\tilde{V}_{tt} = \tilde{V}_{rr}$ 1d wave eqn

$$\begin{cases} \tilde{V} = rV = rG, & t=0 \\ \tilde{V}_t = rH & t=0, \quad \tilde{V}=0, r=0 \end{cases}$$

By d'Alemert formula: for $0 \le r \le t$:

$$\tilde{U}(x;r,t) = \frac{1}{2}\left(\tilde{G}(r+t) - \tilde{G}(t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(s)\,ds$$

$$\tilde{G} = rG, \quad \tilde{H} = rH$$

Now $u(x,t) = \lim_{r\to 0} U(x;r,t) = \lim_{r\to 0} \frac{\tilde{U}(x;r,t)}{r} =$

$$= \lim_{r\to 0} \frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2r} + \lim_{r\to 0} \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(s)\,ds$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$\tilde{G}'(t) = \frac{\partial}{\partial t}\Big(t \fint_{\partial B(x,t)} g(y)\,dS(y)\Big) =$$

$$= \fint_{\partial B(x,t)} g(y)\,dS(y) + t \cdot \underbrace{\frac{\partial}{\partial t}\fint_{\partial B(x,t)} g(y)\,dS(y)}$$

$$\frac{\partial}{\partial t}\fint_{\partial B(0,1)} g(x+ty)\,dS(y) = \fint_{\partial B(0,1)} \nabla g(x+ty)\cdot y\,dS(y) =$$

$$= \fint_{\partial B(x,t)} \nabla g(y)\,\frac{y-x}{t}\,dS(y)$$

$$\Rightarrow u(x,t) = \fint_{\partial B(x,t)} g(y)\,dS(y) + \fint_{\partial B(x,t)} \nabla g(y)\cdot(y-x)\,dS(y) + t\fint_{\partial B(x,t)} h(y)\,dS(y)$$

$$\Rightarrow \left[\, u(x,t) = \fint_{\partial B(x,t)} \left[g + \nabla g\cdot(y-x) + th\right] dS(y) \right]$$

Kirchhoff's formula for 3d wave IVP