Math 678.
Lecture 11.

Weak Max principle for heat eqn (alternative proof)

$U \subset \mathbb{R}^n$ - bdd

$u(x,t)$ - soln to heat eqn $u_t = k\Delta u$, $x \in U$

$U_T = U \times [0,T]$, $\Sigma = \{(x,t) \mid t=0 \text{ or } x \in \partial U\}$

$\Rightarrow$ $\max\limits_{\overline{U}} u = \max\limits_{\Sigma} u$

Proof:

I. Let's show that for any $v(x,t) \in C^2(U) \cap C^1[0,T]$ that has a maximum at an interior pt $(x_0,t_0) \in U$ the following conditions hold:

(a) $\frac{\partial v}{\partial t}(x_0,t_0) = 0$, $\nabla v(x_0,t_0) = 0$

(b) $\Delta v(x_0,t_0) \le 0$

due to (x_0,t_0) being a critical point

For (b), consider $X(s)$ - a parameterized curve going through $X(0) = x_0$

Then $\frac{d^2}{ds^2} v(X(s)) = \frac{d}{ds}\left(\sum_{i=1}^n v_{x_i} \cdot \frac{dX_i}{ds}\right) =$

$$= \sum v_{x_i x_j} \frac{dX_j}{ds} \cdot \frac{dX_i}{ds} + \sum v_{x_i} \frac{d^2 X_i}{ds^2}$$

At (x_0,t_0): $\frac{d^2}{ds^2} v(X(s)) \le 0$ since it's a maximum

$$\left.\frac{d^2 v(X(s))}{ds^2}\right|_{(x_0,t_0)} = \sum_{i,j} v_{x_i x_j} \frac{dX_j}{ds} \cdot \frac{dX_i}{ds} \le 0$$

since $\nabla v(x_0,t_0) = 0$ by (a)

Take $H = \{u_{x_i x_j}\}_{i,j} \le 0 \Rightarrow$ Hessian is seminegative-~~definite~~

$\Rightarrow \lambda_i(H) \le 0$ ← eigenvalues of H.

$$\operatorname{tr} H = \sum_i v_{x_i x_i} = \Delta v \Rightarrow \left[\Delta v(x_0,t_0) \le 0\right]$$
$$\operatorname{tr} H = \sum \lambda_i(H) \le 0$$

$\underline{II}$. $v_\varepsilon(x,t) = u(x,t) + \varepsilon e^{-t}$

where $\varepsilon > 0$, $u_t = k\Delta u$.

To show: $v_\varepsilon(x,t)$ cannot have an interior maximum pt.

Suppose (x_0,t_0) is an interior maximum for $v_\varepsilon(x,t)$, $(x_0,t_0) \notin \Sigma$.

$$\frac{\partial v_\varepsilon}{\partial t} - k\Delta v_\varepsilon = \frac{\partial u}{\partial t} - \varepsilon e^{-t} - k\Delta u = -\varepsilon e^{-t} < 0$$

By part I, $\begin{cases} \frac{\partial v_\varepsilon}{\partial t}(x_0,t_0) = 0, & \nabla v_\varepsilon(x_0,t_0) = 0 \\ \Delta v_\varepsilon(x_0,t_0) \le 0 \end{cases}$

$$\left.\frac{\partial v_\varepsilon}{\partial t}\right|_{(x_0,t_0)} - k\Delta v_\varepsilon\Big|_{(x_0,t_0)} = 0 - k\Delta v_\varepsilon \ge 0 \text{ at } (x_0,t_0)$$

$\Rightarrow$ contradiction $\Rightarrow$ $(x_0,t_0) \in \Sigma$

$\Rightarrow$ $\max_{\bar U} v_\varepsilon = \max_{\Sigma} v_\varepsilon$

As $\varepsilon \to 0$ $\Rightarrow$ $\max_{\bar U} u = \max_{\Sigma} u$ $\square$.

<u>Cauchy problem:</u> IVP for $U = \mathbb{R}^n$

<u>Thm:</u> (MP for Cauchy problem).

$u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

If $u(x,t) \le A e^{a|x|^2}$, $A, a > 0$ $\forall x \in \mathbb{R}^n$, $0 \le t \le T$

Then $\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$

Proof.

1) Assume $4aT < 1 \Rightarrow 4a(T+\varepsilon) < 1$ for some $\varepsilon > 0$

Fix $y \in \mathbb{R}^n$, $\mu > 0$

Define $v(x,t) := u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}$ $\quad x \in \mathbb{R}^n$, $t > 0$

$v_t - \Delta v = 0$ by direct calculation in $\mathbb{R}^n \times (0,T]$

Take some $r > 0$ and consider $U = B(y,r)$

$U_T = B(y,r) \times (0,T]$

By the argument we had for the bdd domains,

$$\max_{\overline{U_T}} v = \max_{\Gamma_T} v$$

2) Take $x \in \mathbb{R}^n$

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} \cdot e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \le u(x,0) = g(x)$$

if $|x-y| = r$, $0 \le t \le T \Rightarrow$

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \le$$

$$\le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \le$$

$|x| - |y| \le |x-y| \le r$

$|x| \le |y| + r$

$$\le Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} \cdot e^{\frac{r^2}{4(T+\varepsilon)}} =$$

$4a(T+\varepsilon) < 1$

$a < \frac{1}{4(T+\varepsilon)} = a + \delta$, $\delta > 0$

$\frac{1}{T+\varepsilon} = 4(a+\delta)$

$$= Ae^{a(|y|+r)^2} - \mu(4(a+\delta))^{n/2} \cdot e^{(a+\delta)\cdot r^2} \le \sup_{\mathbb{R}^n} g$$

for sufficiently large r

$$\Rightarrow \max_{U_T} v = \max_{\Gamma_T} v$$

$v(x,0) \le g(x)$, $x \in \mathbb{R}^n$

$v(x,t) \le \sup_{\mathbb{R}^n} g$, $0 \le t \le T$

$\Rightarrow v(y,t) \le \sup_{\mathbb{R}^n} g$ $\forall y \in \mathbb{R}^n$, $0 \le t \le T$ as long as $\boxed{4aT < 1}$

If we let $\mu \to 0 \Rightarrow \sup_{\mathbb{R}^n \times [0,T]} u(y,t) = \sup_{\mathbb{R}^n} g$

3) If $4aT > 1$

Split $[0,T]$ into intervals $[0,T_1], [T_1, 2T_1]$

$T_1 = \frac{1}{8a}$

$4a \cdot T_1 = \frac{4a}{8a} = \frac{1}{2} < 1 \Rightarrow$ MP is true for each interval, so is true for $[0,T]$.

<u>Uniqueness for Cauchy problem:</u>

$g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0,T])$

there is at most one soln $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$

to the IVP : $\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

satisfying growth condition $|u(x,t)| \le A e^{a|x|^2}$

$A, a > 0$ $x \in \mathbb{R}^n$, $0 \le t \le T$

<u>Pf</u>: Apply MP to $w_1 = u_1 - u_2$ and $w_2 = u_2 - u_1$

Growth condition is essential for uniqueness.

Claim. $\exists$ nonzero smooth function $u \in C^\infty(\mathbb{R}\times[0,\infty))$ satisfying $\begin{cases} u_t - u_{xx} = 0 & , \text{ in } \mathbb{R}\times[0,\infty] \\ u(\cdot,0) = 0 & \text{ on } \mathbb{R} \end{cases}$

__Proof:__

We will construct u s.t. $\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}\times\mathbb{R} \\ u \equiv 0 & \text{for } t<0 \end{cases}$

u - smooth

Consider BVP $\begin{cases} u_t - u_{xx} = 0 \\ u(0,t) = a(t) \\ u_x(0,t) = 0, \end{cases}$ for some ~~reason~~ appropriate $a(t)$.

$u(x,t) = \sum_{k=0}^{\infty} a_k(t)x^k$ - power expansion

Plug this into BVP:

$$\begin{cases} a_{k-2}' = k(k-1)a_k, & k \geq 2 \\ a_0 = a(t) \\ a_1 = 0 \end{cases}$$

$k=2$

$a_0' = 2\cdot 1\cdot a_2$

$a_2 = \frac{1}{2\cdot 1}\cdot a_0'$

$= \frac{1}{2!}a_0'$

$$\Rightarrow \begin{cases} a_{2k}(t) = \frac{1}{(2k)!}a^{(k)}(t) \\ a_{2k+1}(t) = 0 \end{cases}$$

Fix $a(t)$ so that $u(x,t)$ is zero for $t<0$.

$$a(t) = \begin{cases} e^{-1/t^2}, & t>0 \\ 0, & t\leq 0 \end{cases}$$

$u(x,t) = \sum_{k=0}^{\infty} a_k(t)x^k$ - smooth $\Rightarrow$ and $u \equiv 0$ for $t<0$.