Math 678.

Lecture 10..

Thm. (MVT for heat eqn).

$u \in C_1^2(U_T)$ – sdn to heat eqn

Then $$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds$$

Pf: $E(r) = E(0,0;r)$

Last time:

$$\varphi(r) := \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds = \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dy\, ds$$

$$\Rightarrow \varphi'(r) = \frac{1}{r^{n+1}} \Big[\underbrace{\iint_{E(r)} \Big(\sum u_{y_i} y_i \frac{|y|^2}{s^2}\Big) dy\, ds}_{I} + \underbrace{\iint_{E(r)} 2u_s \frac{|y|^2}{s}\, dy\, ds}_{J} \Big]$$

$$\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n \log r$$

1) $\psi = 0$ on $\partial E(r)$

2) $\frac{\partial \psi}{\partial y_i} = \frac{2y_i}{4s} = \frac{y_i}{2s}$

$$J = \frac{1}{r^{n+1}} \iint_{E(r)} 2u_s \frac{|y|^2}{s}\, dy\, ds = \frac{1}{r^{n+1}} \iint_{E(r)} 2u_s \Big(\sum_{i=1}^n \frac{y_i}{2s} \cdot y_i \Big) \cdot 2$$

$$|y|^2 = \sum_{i=1}^n y_i^2$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^n y_i \psi_{y_i}\, dy\, ds =$$

$$y_i \psi_{y_i} = (y_i \psi)_{y_i} - \psi$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \Big(\sum (y_i \psi)_{y_i} - n\psi \Big)\, dy\, ds$$

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n\, u_s \psi\, dy\, ds + \frac{4}{r^{n+1}} \iint_{E(r)} u_s \sum_i (y_i \psi)_{y_i}\, dy\, ds$$

$$= -\frac{1}{r^{n+1}}\Big[\iint_{E(r)} 4n u_s \psi \, dy\, ds + 4\iint_{E(r)} \sum_i (u_s)_{y_i} \cdot y_i \psi \, dy\, ds\Big]$$

since $\psi = 0$ on $\partial E(r)$ by parts in y_i

$$= -\frac{1}{r^{n+1}} \iint_{E(r)} 4n u_s \psi \, dy\, ds + \frac{4}{r^{n+1}}\iint_{E(r)} \sum_i u_{y_i} \cdot y_i \, \psi_s \, dy\, ds$$

by parts in s

$$\psi_s = -\frac{n}{2s} - \frac{|y|^2}{4s^2} \Rightarrow$$

$$= -\frac{1}{r^{n+1}} \iint 4n u_s \psi \, dy\, ds + \frac{4}{r^{n+1}} \iint \sum u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{\cancel{4}s^2}\right) dy\, ds$$

$$= \frac{1}{r^{n+1}}\Big[\iint_{E(r)} -4n u_s \psi \, dy\, ds - \frac{2n}{s}\iint_{E(r)} \sum u_{y_i} y_i \, dy\, ds - \underbrace{\iint_{E(r)} \sum u_{y_i} y_i \frac{|y|^2}{s^2} dy\, ds}_{I}\Big]$$

$$\varphi'(r) = \frac{1}{r^{n+1}}\Big[\iint_{E(r)} -4n u_s \psi \, dy\, ds - \frac{2n}{s}\iint_{E(r)} \sum u_{y_i} y_i \, dy\, ds\Big] =$$

$-u_s = \Delta u$ since u solves heat eqn.

$$= \frac{1}{r^{n+1}}\Big[\iint_{E(r)} 4n \Delta u \, \psi \, dy\, ds - \frac{2n}{s}\iint \sum u_{y_i} y_i \, dy\, ds\Big] =$$

$$= \frac{1}{r^{n+1}} \iint 4n \sum_i u_{y_i} \psi_{y_i} \, dy\, ds - \frac{2n}{s}\iint \sum_i u_{y_i} \cdot y_i \, dy\, ds =$$

$$= \sum_{i=1}^{n} \frac{1}{r^{n+1}}\Big(\iint_{E(r)} 4n u_{y_i} \psi_{y_i} \, dy\, ds - \frac{2n}{s}\iint_{E(r)} u_{y_i} y_i \, dy\, ds\Big)$$

$$\psi_{y_i} = \frac{y_i}{2s} \Rightarrow 4n \cdot u_{y_i} \psi_{y_i} - \frac{2n}{s} u_{y_i} \cdot y_i =$$

$$= 2n \cdot u_{y_i}\left(2 \cdot \frac{y_i}{2s} - \frac{y_i}{s}\right) = 0.$$

$$\Rightarrow \varphi'(r) = 0 \Rightarrow \varphi \equiv \text{const}$$

$$\varphi(r) = \lim_{w \to 0} \varphi(w) = u(0,0)\Big[\lim_{w\to 0} \frac{1}{w^n} \underbrace{\iint_{E(w)} \frac{|y|^2}{s^2} dy\, ds}_{4}\Big] = u(0,0) * 4$$

$$\varphi(r) := \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2} dy\, ds$$

$$\Phi(0-y, 0-s) \geq \frac{1}{r^n}$$

$$E(r) = E(0,0;r) = \{(y,s) \mid s \leq 0, \Phi \geq \frac{1}{r^n}\}$$

SMP for heat equation.

Let $u \in C_1^2(U_T) \cap C(\overline{U_T})$ solves heat eqn in U_T.

$\Rightarrow$ (i) $\max\limits_{\overline{U_T}} u = \max\limits_{\Gamma_T} u \qquad \Gamma_T \leftarrow T_T = \partial(\overline{U_T})$ (MP)

(ii) if U - connected and $\exists (x_0, t_0) \in U_T$ s.t.

$u(x_0, t_0) = \max\limits_{\overline{U_T}} u$

then $u \equiv$ const on $\overline{U}_{t_0}$. (SMP)

<u>Proof</u>: (ii) Take $(x_0, t_0) \in U_T$ for which $u(x_0, t_0) = \max\limits_{U_T} u$

denote $M := \max\limits_{\overline{U_T}} u$

Fix some $r > 0$ s.t. $E(x_0, t_0; r) \subset U_T$

By MVT: $M = u(x_0, t_0) = \frac{1}{4r^n} \iint\limits_{E(x_0, t_0; r)} \underbrace{u(y,s)}_{\leq M} \frac{|x_0 - y|^2}{(t_0 - s)^2} \, dy ds$

$$\frac{1}{4r^n} \iint\limits_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{|t_0 - s|^2} \, dy ds = 1$$

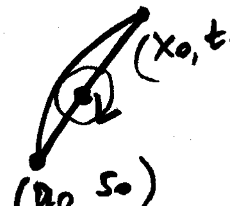
Equality will occur only if $u \equiv M$ on $E(x_0, t_0; r)$

$\Rightarrow u(y, s) = M \quad \forall (y, s) \in E(x_0, t_0, r)$

If U connected then there is a line segment L connecting (x_0, t_0) with $(y_0, s_0) \in U_T$ with $s_0 < t_0$.

Consider $r_0 := \min\{ s \geq s_0 \mid u(x,t) = M \ \forall (x,t) \in L, \ s \leq t \leq t_0 \}$

attained since u is continuous.



Suppose $r_0 > s_0$

Then $u(z_0, r_0) = M$ for $(z_0, r_0) \in L \cap U_T$

$\Rightarrow u \equiv M$ on $E(z_0, r_0; r)$ for some sufficiently small $r > 0$.

$E(z_0, r_0; r)$ contains $L \cap \{r_0 - \sigma \le t \le r_0\}$
$\Rightarrow$ this contradicts the $\quad \sigma > 0$
fact that r_0 was a minimum. $\Rightarrow r_0 = s_0$.
Now $u(x,t) \equiv$ Const $= M$ on L.

Now take a sequence of points
(x_i, t_i) such that $\overline{x_{i-1} x_i} \in U$ and
on segment (x_{i-1}, t_{i-1}) to $(x_i, t_i) \in U_T$.
$u(x,t) = M \Rightarrow u \equiv M$ everywhere
on $\overline{U_{t_0}}$.

Corollaries.

1) U-connected, $u \in C_1^2(U_T) \cap C(\overline{U_T})$
satisfies $\begin{cases} u_t - \Delta u = 0 & U_T \\ u = 0, & \partial U \times [0,T] \\ u = g & U \times \{t=0\}, \ \underline{g \ge 0} \end{cases}$

If $g > 0$ somewhere on U then $u > 0$ everywhere in U_T

2) Infinite propagation speed for disturbance.

3) If $f \in C(U_T)$, $g \in C(\Gamma_T) \Rightarrow$ there is at most one solution to $\begin{cases} u_t - \Delta u = f, & U_T \\ u = g & \Gamma_T \end{cases}$
IBVP

Pf: $w_1 = u - \tilde{u}$ where $u, \tilde{u}$ both solve this IBVP
$w_2 = \tilde{u} - u \Rightarrow$ Both solve $\begin{cases} w_{i\,t} - \Delta w_i = 0 \\ w = 0 \quad \Gamma_T \\ w = 0 \quad \text{on } t=0 \end{cases}$
$\Rightarrow \max\limits_{\overline{U_T}} w_1 = \max\limits_{\overline{U_T}} (u - \tilde{u}) = \max\limits_{\Gamma_T}(w_1) = 0$
$\max\limits_{\overline{U_T}} w_2 = \max\limits_{\overline{U_T}} (\tilde{u} - u) = \max\limits_{\Gamma_T} w_2 = 0$

$\Rightarrow u \equiv \tilde{u}$ on $\overline{U_T}$.