Math 678.
Lecture 1.

Def. $F(D^k u(x), D^{k-1}u(x) \ldots Du(x), u(x), x) = 0 \quad (1)$

$x \in U \subset \mathbb{R}^n$

is called a k-th order PDE, where

$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ given

$u : U \to \mathbb{R}$ is the unknown.

Types of PDE:

(i) (1) is a linear PDE if it has the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

$D^\alpha u = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$
multiindex notation

(ii) (1) is a semilinear PDE if it has the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0$$

(iii) (1) is a quasilinear PDE if it has the form

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \ldots Du, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots Du, u, x) = 0$$

(iv) (1) is fully nonlinear if it involved nonlinear highest order derivative terms.

Def. $\vec{F}(D^k\vec{u}(x), \ldots D\vec{u}(x), \vec{u}(x), x) = 0$, $x \in U$
is k-th order system of PDEs
where $\vec{F}: \mathbb{R}^{mn^k} \times \ldots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$
$\vec{u}: U \to \mathbb{R}^m$, $u = (u^1, \ldots u^m)$ - unknown

Examples.

1) $\Delta u = 0$ Laplace's eqn
$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$
$-\Delta u = f$ Poisson's eqn, $f \neq 0$.

2) $-\Delta u = \lambda u$ - Eigenvalue eqn (Helmholtz)

3) $u_t + \sum_{i=1}^n b^i u_{x_i} = 0$ transport equation

4) $u_t - \sum_{i=1}^n (b^i u)_{x_i} = 0$ Liouville's eqn

5) $u_t - \Delta u = 0$ heat (diffusion) eqn
$u_t - \Delta u = f(u)$ reaction-diffusion eqn

6) $u_t - \sum_{i,j=1}^n (a^{ij} u)_{x_i x_j} - \sum_{i=1}^n (b^i u)_{x_i} = 0$ Fokker-Planck eqn

7) $u_{tt} - \Delta u = 0$ wave eqn

Solving PDE.

Well-posed problem:
1) has a solution
2) has a unique solution
3) solution depends continuously on the data (given)

Classical soln of a k-th order PDE is a function $u \in C^k$ ~~and~~ satisfying (1).

Sometimes classical solutions do not exist, e.g. $u_t + F(u)_x = 0$ shock wave eqn then you resort to a weak solution

If the assumption of $u \in C^k$ is weakened, the problem can still be well-posed.

Existence problem: proving well-posedness in some appropriate class of weak solutions.

Regularity problem: proving the weak solution is actually smooth to be called a classical soln.

Overview.

1. Representation formulas for solutions.
2. Linear PDE theory
3. Nonlinear PDE theory.

§2. Important Linear PDEs.

1. Transport equation:

$$u_t + \vec{b} \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \qquad (1)$$

$u = u(x,t)$

$Du = D_x u = (u_{x_1}, \dots u_{x_n})$ - gradient wrt x.

Assume $u(x,t)$ - smooth function

Fix a point $(x,t) \in \mathbb{R}^n \times (0,\infty)$

$z(s) := u(x+sb, t+s)$, $s \in \mathbb{R}$

$$\frac{dz}{ds} = Du(x+sb, t+s)\cdot b + u_t(x+sb, t+s) = 0$$

(since $Du \cdot b + u_t = 0$)

$\Rightarrow z(\cdot) \equiv \text{const}$ wrt s

This means that for any (x,t), u is constant along the line through (x,t) in the direction of $(\vec{b}, 1) \in \mathbb{R}^{n+1}$.

We know that if the value of u is known anywhere on this line, we have solved the problem.

<u>IVP</u>: $\begin{cases} u_t + b\cdot Du = 0 & \mathbb{R}^n \times (0,\infty) \\ u = g & \text{if } t=0, \text{ i.e. on } \mathbb{R}^n \times \{t=0\} \end{cases}$

The line through (x,t) with direction $(b,1)$ is represented by $(x+sb, t+s)$, $s \in \mathbb{R}$.

This line hits the plane $\Gamma := \mathbb{R}^n \times \{t=0\}$ when $s=-t$ at the point $(x-tb, 0)$.

Here $u(x-tb, 0) = g(x-tb)$ is known.

Now $u(x,t) = g(x-tb)$, $x \in \mathbb{R}^n$, $t \geq 0$

So assuming $u(x,t)$ is sufficiently regular, this gives the solution to IVP.

Nonhomogeneous problem:

$$\begin{cases} u_t + b\cdot Du = f & , \ \mathbb{R}^n \times (0,\infty) \\ u = g & , \ \mathbb{R}^n \times \{t=0\} \end{cases}$$

Fix $(x,t) \in \mathbb{R}^{n+1}$

Set $z(s) := u(x+sb, t+s)$, $s \in \mathbb{R}$

$$\frac{dz}{ds} = \dot{z}(s) = Du(x+sb, t+s)\cdot b + u_t(x+sb, t+s)$$

$$= f(x+sb, t+s) \quad \text{from the original eqn.}$$

Before we had $u(x,t) = g(x-tb)$

Now look at $u(x,t) - g(x-tb) =$

$$= z(0) - z(-t) = \int_{-t}^{0} \dot{z}(s)\,ds =$$

$$= \int_{-t}^{0} f(x+sb, t+s)\,ds = \int_{0}^{t} f(x+(s-t)b, s)\,ds$$

$$\Rightarrow \boxed{u(x,t) = g(x-tb) + \int_{0}^{t} f(x+(s-t)b, s)\,ds ,}$$