Math 678. Fall 2011. Final Exam.
Due Wednesday 12/14/11 by 5pm

Solutions should represent individual work, with all necessary details. Only facts discussed in class or given in the main textbook can be used without proof (except the facts known from calculus).

Problem 1. Consider

$$\begin{cases} u_{xx} + u_{yy} &= 0, & \text{in } x^2 + y^2 < 1 \\ u(x,y) &= 1 + 2x^4, & \text{on } x^2 + y^2 = 1 \end{cases}$$

(a) Prove that $u(x,y) > 0$
(b) Compute $u(0,0)$ by using Poisson formula

Problem 2. Formulate Duhamel's principle and solve the initial value problem

$$\begin{cases} u_t &= ku_{xx} + f(x,t), \quad x > 0, t > 0 \\ u(x,0) &= 0, x > 0 \\ u(0,t) &= 0, t > 0 \end{cases}$$

Problem 3. Find the bounded solution to the Neumann problem by Fourier transform method:

$$\begin{cases} u_{xx} + u_{yy} &= 0, \quad x \in \mathbb{R}, y > 0 \\ u_y(x,0) &= g(x), \quad x \in \mathbb{R} \end{cases}$$

Hint: consider a change of variables $v = u_y$.

Problem 4. Find a smooth solution of $u_t + (u_x)^4 = 0, x \in \mathbb{R}, t > 0$ with initial data $u(x,0) = \dfrac{3}{4}x^{4/3}$.

Problem 5. Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve

$$\begin{cases} u_{tt} - u_{xx} &= 0, \text{ in } \mathbb{R} \times (0,\infty) \\ u = g, u_t &= h, \text{ on } \mathbb{R} \times t = 0 \end{cases}$$

with g, h having compact support. The integral $k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ is the kinetic energy, and $p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$ is the potential energy. Show that:
(a) $k(t) + p(t)$ is constant for all t
(b) $k(t) = p(t)$ for sufficiently large t.
This is the equipartition of energy principle for the wave equation.