1. (4.1, # 9) Prove that $\det(AB) = \det(A) \cdot \det(B)$ for any 2×2 matrices A and B.

   Proof: Let
   $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$
   Then $\det(A) = ad - bc$ and $\det(B) = xw - yz$, so their product is
   $$\det(A) \cdot \det(B) = (ad - bc)(xw - yz) = adxw + bdyz - adyz - bdxw.$$
   Then
   $$AB = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$$
   Thus $\det(AB) = (ax + bz)(cy + dw) - (ay + bw)(cx + dz)$
   $$= axcy + axdw + bzcy + bzdw - aycx - aydz - bwcx - bwdz.$$
   Then the first and last positive terms cancel with the first and last negative terms, leaving $axdw + bzcy - aydz - bwcx$, which is the product of the determinants given above.

2. (4.2, # 25) Prove that $\det(kA) = k^n \det(A)$ for any $n \times n$ matrix A.

   Proof: Multiplying the matrix A by k is equivalent to multiplying each row by k. Since the det function is n-linear, we can pull the factor k from each row, one at time and put it in front of $\det(A)$. Thus
   $$det(kA) = det \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{pmatrix} = kdet \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & & & \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{pmatrix} = \cdots$$
   Thus the result is clear..

3. (Sec. 4.3, # 12) A square matrix Q is called *orthogonal* if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

   Proof: $1 = \mathrm{Det}(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)\det(Q)$. Thus $\det(Q)^2 = 1$. Thus $\det(Q) = 1$ or -1.