1. (1.6, # 30) Let $V = M_{2\times 2}(F)$ and let W_1 and W_2 be the subsets of V as defined.

Proof: We show that W_1 is a subspace. Let $X, Y \in W_1$. Then say

$$X = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \text{ and } Y = \begin{pmatrix} e & f \\ g & e \end{pmatrix}$$

Then $X + Y = \begin{pmatrix} a+e & b+f \\ c+g & a+e \end{pmatrix}$ and since the entries on the main diagonal ar equal, this matrix is in W_1. Similarly it is easy to see that αA is in W_1 for $\alpha \in F$. Hence W_1 is a subspace. The same thing works for W_2.

To find a basis for W_1, we note that a, b and c are free variables. So here is a candidate for a basis:

$$\left\{ u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Clearly $\begin{pmatrix} a & b \\ c & a \end{pmatrix} = au_1 + bu_2 + cu_3$. So the set spans W_1. To check liner independence, one writes:

$$\alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{pmatrix} = \mathbf{0}.$$

Clearly this implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus $\{u_1, u_2, u_3\}$ is a linearly independent set. Hence it is a basis. Thus $\dim(W_1) = 3$.

It is also direct to see that

$$\left\{ v_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

is a basis of W_2. Thus W_2 has dimension 2 (there are two free variables in describing elements of W_2).

Since $W_1 + W_2$ is bigger than W_1, it follows that $\dim W_1 + W_2 > 3$. But this is a subspace of V which has dimension 4. Hence $W_1 + W_2$ has dimenion 4. Finally to describe the intersection of the two spaces, notice that for a matrix to be in the intersection entire on the main diagonal must be equal to be in W_1, while they must be 0 to also be in W_2. Therefore we have

$$W_1 \cap W_2 = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : a, b \in F \right\}.$$

Clearly there is only one free variable. Hence the space has dimension 1.

2. (Sec. 2.1, # 14a) The next solution is included just to show you how it is done. Let $T : V \longrightarrow W$ be linear. Show that T is injective iff T carries linearly independent sets to linearly independent sets.

   Proof: Let T be injective. Let $u_1, \ldots, u_m$ be a linearly independent set in V. We want to show that $T(u_1), \ldots, T(u_m)$ is a linearly independent set (in W). Suppose

   $$a_1 T(u_1) + \ldots + a_m T(u_m) = \mathbf{0}$$

   Then by the linearity of T we have $T(a_1 u_1 + \ldots + a_m u_m) = \mathbf{0}$. But T is injective, hence $a_1 u_1 + \ldots + a_m u_m = \mathbf{0}$. But this original set is linearly independent. Hence all $a_i = 0$ and we are done in this direction.

   Suppose that T carries linearly independent sets to linearly independent sets. Show that T is injective. It suffices to show that if u is a nonzero vector in V, then $T(u) \neq \mathbf{0}$. But any set with one nonzero vector is a linearly independent set. Thus by assumption, $\{T(u)\}$ is a linearly independent set and so $T(u) \neq \mathbf{0}$. This proves that the null space of T is trivial.

   (Sec. 2.1, # 14b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent iff $T(S)$ is linearly independent.

   Proof: Suppose that T is one-to-one. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent set in V. Then by part (a) above, $T(S)$ is linearly independent.

   Conversely, assume that $T(S)$ is linearly independent. We want to show that S is linearly independent (we don't need the fact that T is one-to-one for this). Suppose that

   $$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = \mathbf{0}.$$

   Then $\mathbf{0} = T(a_1 u_1 + a_2 u_2 \ldots + a_n u_n) = a_1 T(u_1) + a_2 T(u_2) + \ldots + a_n T(u_n)$. But by assumption the set $\{T(u_i)\}_{i=1,2,\ldots,n}$ is a linearly independent set. Hence $a_1 = a_2 = \ldots = a_n = 0$. Thus S is a linearly independent set.

3. (Sec. 2.1, # 17) Let V and W be finite dimensional vector spaces and let $T :\rightarrow W$ be linear.

   (a) Prove that of $\dim(V) < \dim(W)$, then T cannot be onto

   (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

   Proof: The result we will use for both parts is Theorem 2.3 which states that $\dim(V) = \text{nullity}(T) + \text{rank}(T)$. Also note that for any transformation T, it is onto iff $\text{rank}(T) = \dim(W)$ and it is one-to-one iff $\text{nullity}(T) = 0$.

   (a) Suppose that $\dim(V) < \dim(W)$. Note that by Theorem 2.3, $\text{rank}(T) \leq \dim(V)$ for any linear transformation, while by assumption $\dim(V) < \dim(W)$. Hence $\text{rank}(T) < \dim(W)$ and so T is not onto.

   (b) By the theorem we have $\dim(V) = \text{nullity}(T) + \text{rank}(T)$ or $\dim(V) - \text{nullity}(T) = \text{rank}(T)$. Since the range space is a subspace of W, we have that $\text{rank}(T) \leq \dim(W) < \dim(V)$. Thus $\text{nullity}(T) > 0$ and so T is not one-to-one.