NAME (print): KEY

Math 322 Fall 2015—Exam 2

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[24pts] 1. Fill in the blanks with ALWAYS, SOMETIMES or NEVER.

(a) Any invertible matrix can Always be written as a product of elementary matrices.

(b) Let A be a 3×2 matrix. If there are two linearly independent rows of A, then Never $\text{rank}(A) = 1$.

(c) Let T be an operator on V where $\dim(V) = 4$. Suppose that T has two eigenvalues λ_1 and λ_2. If T is diagonalizable and if $\dim(E_{\lambda_1}) = 2$, then $\dim(E_{\lambda_2})$ Alwasy equals 2.

(d) Let V be a finite dimensional vector space. If T is a one to one linear operator on V, then T is Always onto.

(e) If A is a 3×3 matrix with exactly two distinct eigenvalues, then A is Sometimes diagonalizable.

(f) Let A be an $n \times n$ matrix over the reals such that $Ax = \mathbf{0}$ has a nontrivial solution. If $Ax = \mathbf{b}$ has a solution, then it Sometimes has infinitely many solutions.

[10pts] 2. Show that if λ is any real number, then the matrix $\begin{pmatrix} \lambda & 0 & -1 \\ 0 & 2 & 4 \\ 1 & -1 & \lambda \end{pmatrix}$ is invertible.

Compute the determinant and see where it can be zero. (Expand along first row.) The determinant of the above matrix is $\lambda \cdot (2\lambda + 4) + (-1) \cdot (-2) = 2\lambda^2 + 4\lambda + 2 = 2(\lambda^2 + 2\lambda + 2 = 2(\lambda + 1)^2$. This has exactly one root $\lambda = -1$. Thus the matrix is invertible for all values of λ, except $\lambda = -1$.

[10pts] 3. Show that the function $\delta : M_{2\times 2} \to F$ given by $\delta(A) = A_{11} + A_{12}$ is not 2-linear.

We will show that it is not linear in the second row.

$$\delta\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 1 + 1 = 2$$

But

$$2\delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2(1+1) = 4$$

[16pts] 4. Determine if the following linear operators are diagonalizable. If they are, find a diagonal matrix representation for the operator.

(a) T is multiplication of $\mathbb{R}^2$ by the matrix $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

First find char(A). Char(A) $= \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 2$. Since the characteristic polynomial has no real roots, it is not diagonalizable. (if you used complex roots, I accepted that).

(b) Where T is the operator on $\mathbb{R}^3$ defined by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + b \\ 2c \\ 2b \end{pmatrix}$

The matrix of T with respect to the standard basis is

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

You first find the characteristic polynomial of this matrix. To do this, subtract λI and compute determinant by expanding along first column (this is easiest). You get $(3-\lambda)(\lambda^2-4)$. This has three roots: 3, 2 and -2. Since it is a 3×3 matrix it must be diagonal. In particular it is similar to

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Note there are other choices for D.

[10pts] 5. Let A and B be $n \times n$ matrices over the reals such that $AB = -BA$. Show that at least one of A or B is not invertible. (Hint: Consider determinants.)

Proof. Then $\det(AB) = \det(-BA) = (-1)^n det(BA) = -det(BA)$, since n is odd. But $\det(AB) = det(A)det(B) = det(B)det(A)$. Thus $det(AB) = -det(BA) = -det(AB)$. The only real number that is equal to its own negative is 0. Thus $det(AB) = 0$, and so AB is not invertible. But if both A and B are invertible, then so is AB, which is impossible.

[10pts] 6. Let T be a linear operator on the vector space V. Let u and v be nonzero elements of V such that $T(u) = u$ and $T(v) = 2v$. Prove that $\{u, v\}$ is a linearly independent set.

If they are not linearly independent, then one is a multiple of the other, say $u = \lambda v$. Thus $u = T(u) = \lambda T(v) = \lambda(2v) = 2\lambda v$. Which contradicts $u = \lambda v$.

[10pts] 7. Let $V = \mathcal{P}_3(\mathbb{R})$ and let T be the linear operator on V given by $T(f(x)) = xf'(x)$. Determine if the subspace $W = \mathcal{P}_2(\mathbb{R})$ is T-invariant.

Note $W = \{f(x) = ax^2 + bx + c : a, b, c \in \mathbb{R}\}$. Then $T(f(x)) = x(2ax + b) = 2ax^2 + bx$, which is clearly in W. Thus W is T-invariant.

[10pts] 8. Let T be an operator on a two-dimensional space V. Prove that either V is a T-cyclic subspace of itself or $T = c1_V$ for some scalar c.

Let $\mathbf{0} \neq v \in V$. If $T(v)$ is not a multiple of v, then $v, T(v)$ is a linearly independent set. Hence it spans V, and thus $V = Span\{v, T(v)\}$. Hence V is T-cyclic. Now suppose that $T(v) = cv$ for some scalar c (the only other possibility). Let $u \in V$ be a vector that is not a multiple of v. Thus $\{v, u\}$ is a basis of V. If $T(u)$ is not a multiple of u, then as before V is T-cyclic and again we are done. If $T(u) = cu$ (the same c as before), then $T = c1_V$ and we are done. The only other choice is that $T(u) = du$ for some scalar $d \neq c$. Now consider $v + u$. If this is an eigenvector of T, then it would have to to a different eigenvalue then c or d. This would be a third eigenvalue of T, which is not possible, since the domain has dimension 2. Hence $T(v + u)$ is not a multiple of $v + u$, and so as before $Span\{v + u, T(v + u)\} = V$. So V is T-cyclic.