NAME (print): ____________________

Math 322 Fall 2015—Exam 1

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Work carefully and neatly, remember I cannot grade what I cannot read. Please write down the problems in order. Show all relevant work! Print your name at the top! Note this exam is worth only 98 points!

[24pts] 1. Fill in the blanks with NEVER, SOMETIMES, or ALWAYS.

(a) A non-empty subset of a linearly independent set is Always linearly independent.

(b) Let $T : V \to V$ be linear. If dim $\mathcal{R}(T) =$ dim V, then T is Always one-to-one.

(c) If V has dimension 4 and if S spans V, then S Always has at least 4 elements.

(d) If S spans V and if $S \subset S'$ properly, then S' is Never a basis of V.

(e) Let V be a finite dimensional space and let S be a linearly independent subset of V. Then there Always exists a basis S' of V that contains S.

(f) Let A be a 3×4 matrix. Then rank(A) Sometimes equals 3.

(g) If a 5×5 matrix A has rank 4, then A is Never a product of elementary matrices.

(h) If the matrices A and B can be row reduced to the same matrix, then A and B Always have the same rank.

[12pts] 2. Let $T : V \to W$ be a linear transformation between vector spaces V and W. Prove that T is one-to-one if and only if $\mathcal{N}(T) = \{\mathbf{0}\}$.

First assume that T is one-to-one. We know that $T(\mathbf{0}) = \mathbf{0}$. By the one-to-one property, nothing else can map to $\mathbf{0}$. Thus $\mathcal{N}(T) = \{\mathbf{0}\}$.

Conversely assume that $\mathcal{N}(T) = \{\mathbf{0}\}$. Let $T(u) = T(v)$. Then $T(u - v) = T(u) - T(v) = \mathbf{0}$. Thus $u - v \in \mathcal{N}(T)$. Hence $u - v = \mathbf{0}$ or $u = v$. Thus T is one-to-one.

[12pts] 3. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-to-one?

We want to write $(2, 3)$ as a linear combination of $(1, 0)$ and $(1, 1)$. Because of the 0 in the second coordinate of the first vector, it easily follows that

$$(2, 3) = -(1, 0) + 3(1, 1).$$

Hence

$$T(2, 3) = T(-(1, 0)+3(1, 1)) = -T(1, 0)+3T(1, 1) = -(1, 4)+3(2, 5) = (5, 11).$$

By the way T is one-to-one. The vectors $(1, 4)$ and $(2, 5)$ are linearly independent, and so they are a basis of $\mathbb{R}^2$. Hence T is onto, therefore by the dimension Theorem T is one-to-one.

[12pts] 4. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be defined by $T(f(x)) = f''(x) + xf'(x) - 3f(x)$. Then T is linear (you don't have to prove this). Find the matrix of T with respect to the basis $\beta = \{1, x, x^2\}$. Determine if T is invertible.

We have to compute T of each vector in β. Then

$$T(1) = 0 + x \cdot 0 - 3(1) = -3.$$

$$T(x) = 0 + x(1) - 3(x) = -2x \text{ and}$$
$$T(x^2) = 2 + x(2x) - 3(x^2) = 2 - x^2.$$

Thus

$$[T]_\beta = \begin{pmatrix} -3 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is easy to row reduce this matrix, in particular there will be a pivot in each column. Thus the matrix is invertible. Hence T is invertible.

[12pts] 5. Let V and W be vector spaces and let $T : V \to W$ be a one-to-one linear transformation. Show that if S is a linearly independent subset of V, then $T(S)$ is a linearly independent subset of W.

Suppose that there exist vectors $v_1, v_2, \dots, v_n \in S$ and scalars $a_1, a_2, \dots, a_n$ such that

$$a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = \mathbf{0}.$$

We must show that all the a_i equal 0. Then by linearity we have $T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = \mathbf{0}$. Hence $a_1v_1 + a_2v_2 + \cdots + a_nv_n \in \mathcal{N}(T)$. But T is one-to-one. Thus $\mathcal{N}(T) = \mathbf{0}$, i.e., $a_1v_1 + a_2v_2 + \cdots + a_nv_n = \mathbf{0}$. But the v_i's are linearly independent. Hence $a_1 = a_2 = \cdots = a_n = 0$. Which proves that $T(S)$ is linearly independent.

[12pts] 6. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ (the zero function) if and only if $\mathcal{R}(T) \subseteq \mathcal{N}(T)$.

First suppose that $T^2 = T_0$. That means for all $v \in V, T \circ T(v) = \mathbf{0}$. Or in other words $T(T(v)) = \mathbf{0}$. Hence $T(v) \in \mathcal{N}(T)$. Thus $\mathcal{R}(T) \subseteq \mathcal{N}(T)$.

Conversely suppose that $\mathcal{R}(T) \subseteq \mathcal{N}(T)$. Hence for all $v \in V$, $T(v) \in \mathcal{N}(T)$. Hence $T^2(v) = T(T(v)) = \mathbf{0}$, i.e., $T^2 = T_0$.

[12pts] 7. Let V be a vector space. Recall that V is said to be the direct sum of the subspaces W_1 and W_2, if $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\mathbf{0}\}$. Show that if V is the direct sum of W_1 and W_2 and if β_1 and β_2 are bases of W_1 and W_2 respectively, then $\beta = \beta_1 \cup \beta_2$ is a basis of V.

We have to show that β spans V and is linearly independent. Let $\beta_1 = \{u_1, u_2, \dots, u_t\}$ and $\beta_2 = \{v_1, v_2, \dots, v_s\}$.

Spans: Let $x \in V$. Since $W_1 + W_2 = V$, $x = u + v$ where $u \in W_1$ and $v \in W_2$. Since β_1 is a basis of W_1 we have

$$u = a_1u_1 + a_2u_2 + \dots a_tu_t.$$

Similarly, v is a linear combination of the v_j. Hence x is is a linear combination of vectors in $\beta_1 \cup \beta_2 = \beta$.

Linearly independent: Suppose that

$$a_1u_1 + a_2u_2 + \dots + a_tu_t + b_1v_1 + \dots + b_sv_s = \mathbf{0}.$$

Then we have

$$a_1u_1 + a_2u_2 + \dots + a_tu_t = -b_1v_1 - \dots - b_sv_s \in W_1 \cap W_2 = \{\mathbf{0}\}$$

Hence $a_1u_1 + a_2u_2 + \dots + a_tu_t = \mathbf{0}$. Since β_1 is a linearly independent set, $a_i = 0$ all i. Similarly, $b_j = 0$ all j. Hence β is a linearly independent set.