NAME (print legibly - 2pts): ______________________ KEY

Math 321 Spring 2016—Exam 2

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Please work neatly and answer the problems in the space provided. Remember, I cannot grade what I cannot read! Print your name at the top.

[28pts] 1. The following questions are short answer, requiring only a brief justification (but there must be some!):

(a) Suppose that φ is a homomorphism from Z_{30} to Z_{30} where Ker $\varphi = \{0, 10, 20\}$. If $\varphi(23) = 9$, determine all elements that map to 9.

$23+$ ker $\varphi = \varphi^{-1}(9)$. Then
$23+$ ker $\varphi = \{23, 33 = 3, 43 = 13\} = \{23, 3, 13\}$.

(b) Let H and K be subgroups of a group G such that $|H| = 30$ and $|K| = 42$. What are the possible orders of $H \cap K$?

By Lagrange, $|H \cap K|$ divides $|H| = 30$ and $|K| = 42$. The common divisors of these numbers are 1, 2, 6.

(c) What is the order of the element $4 + \langle 6 \rangle$ in the factor group $Z_{24}/\langle 6 \rangle$?

We want the smallest integer n such that $n \cdot 4$ is in the subgroup of Z_{24} generated by 6. Clearly $\langle 6 \rangle = \{0, 6, 12, 18\}$. So the smallest multiple of 4 that is in that set is 3. Hence the order of the element is 3.

(d) Let $\varphi : Z_6 \longrightarrow S_5$ be a group homomorphism such that $\varphi(1) = (1, 2, 3)(4, 5)$. What is $\varphi(3)$?

Since $3 = 3 \cdot 1$, and since φ is a group homomorphism, we have $\varphi(3) = [(1, 2, 3)(4, 5)]^3$. Since the permutations are disjoint, we have $[(1, 2, 3)(4, 5)]^3 = (1, 2, 3)^3(4, 5)^3 = (1)(4, 5)$.

[16pts] 2. In the group $\mathbf{Z}_{10} \times S_4$, let a be the element $(3, (1\,2\,4))$.

(a) Find $|a|$.

In $\mathbf{Z}_{10}$, $|3| = 10$, since 3 is relatively prime to 10, while the order of $(1\,2\,4)$ is three. Thus the order of a is lcm(10,3) = 30.

(b) Find a^{-1}.

To compute a^{-1}, we compute the inverse of each coordinate. Thus $a^{-1} = (7, (1\,4\,2)$.

[10pt] 3. Show that any group of order 12 has an element of order 2.

By Lagrange, every element of G must have order either 1, 2, 3, 4, 6, or 12. If there exists an element $a \in G$ that has order 2, 4, 6 or 12, then clearly the appropriate power of a will have order 2 and we are done. Hence we assume that every element (other than the identity) has order 3 and arrive a contradiction.

Let a have order 3, and let $H_1 = \langle a \rangle$. So $|H_1| = 3$. Pick $b \in G \setminus H_1$ and let $H_2 = \langle b \rangle$. Then $H_1 \cap H_2$ has only the identity element. Hence $H_1 \cup H_2$ has 5 elements. Now pick $c \in G \setminus (H_1 \cup H_2)$ and let $H_3 = \langle c \rangle$. In this way we keep adding 2 new elements. Since we started with 3, we can never reach exactly 12 elements. Hence we have the desired contradiction.

[10pts] 4. Explain why the groups D_4 (the group of symmetries of the square) and $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quarternions) are not isomorphic.

Note that D_4 has exactly 2 elements of order 4, namely R_{90} and R_{270}. On the other hand Q has 6 elements of order 2, namely $\{\pm i, \pm j, \pm k\}$. So the groups cannot be isomorphic.

[16pts] 5. Let $G = S_3 \times \mathbf{Z}_2$.

(a) Let H be the subgroup of G generated by $((1\,2), 1)$. List the elements of H. (Caution: The first coordinate is multiplicative, while the second is additive.)

Since both coordinates of this element have order 2, the element itself has order 2. Thus

$$H = \{((1), 0), ((1\,2), 1)\}$$

(b) Write the left cosets of H in G.

There must be $12/2 = 6$ distinct cosets. Besides H, they are
$((1\,2), 0) * H = \{(1,\ 2), 0), ((1), 1)\}$,
$((1\,3), 0) * H = \{(1\,3), 0), ((1\,2\,3), 1)\}$,
$((1\,3), 1) * H = \{((1\,3), 1), ((1\,2\,3), 0)\}$,
$((2\,3), 0) * H = \{((2\,3), 0), ((1\,3\,2), 1)\}$,
$((2\,3), 1) * H = \{((2\,3), 1), ((1\,3\,2), 0)\}$

[18pts] 6. Let R^* denote the group of non-zero reals under multiplication.

(a) Let $\mathbb{R}^+$ denote the subgroup of $\mathbb{R}^*$ consisting of the positive reals. Show that $\mathbb{R}^+$ has index 2 in $\mathbb{R}^*$ (i.e., there are exactly two cosets of $\mathbb{R}^+$ in $\mathbb{R}^*$).

One coset is $R^\#$, another is $(-1)R^\#$. Clearly the union of these two sets is all of R^*. Hence there are precisely two cosets, so the subgroup has index 2.

(b) Let H be any subgroup of $\mathbb{R}^*$ that has index 2. Show that for any $a \in \mathbb{R}^*$, $a^2 \in H$. (Hint: What is the order of the quotient group G/H, and how does that help?)

Clearly H is a normal subgroup, since the group is abelian. Thus the quotient group R^*/H has order two. This means that any coset squared is the identity coset, i.e., $(aH)^2 = H$ or $a^2 \in H$ for all elements $a \in R^*$.

(c) Deduce that any subgroup of $\mathbb{R}^*$ having index 2 must contain R^+ and hence must equal R^+.

Let H be any subgroup of index 2. By part (b) the square of any element is in H. Since every positive real is the square of some number ($a = (\sqrt{a})^2$), we see that H must contain R^+. But this group already has index 2 and any bigger subgroup would have index 1, i.e., would equal $\mathbb{R}^*$, so $H = R^+$