NAME (print): Answer Key

Math 321 Spring 2016—Exam 1

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Please work neatly and write complete answers in the space provided. (If you cannot fit it into that space, point me to your work.) Remember I cannot grade what I cannot read!

[40pts] 1. The following are short answer, requiring only a brief response.

(a) If a, b, and c are elements of a group and $|a| = 6, |b| = 7$, express $(a^4c^{-2}b^4)^{-1}$ without using negative exponents.

Since $a^6 = e$, $(a^4)^{-1} = a^2$. Similarly $(b^4)^{-1} = b^3$. Thus $(a^4c^{-2}b^4)^{-1} = (b^4)^{-1}c^2(a^4)^{-1} = b^3c^2a^2$

(b) Let $a, b \in G$, G any group and let H be a subgroup of $\langle a\rangle \cap \langle b\rangle$. If $|a| = 20$ and $|b| = 12$, what are the possible orders of H? (Remember $\langle x\rangle$ is always cyclic.)

Note that H is a subgroup of two cyclic subgroups, $\langle a\rangle$ and $\langle b\rangle$. Hence the order of H must divide 4 and 20. Hence $|H| = 1$ or 2 or 4.

(c) Find the order of the element 10 in the group $\mathbb{Z}_{15}$.

Can use brute force: $10 + 10 = 20 = 5 \bmod 15$ and $10 + 10 + 10 = 0 \bmod 15$. Thus $|10| = 3$.

(d) Determine if $\alpha = (12)(134)(152)$ is an even or an odd permutation.

Note that α is the product of an odd with two even permutations. Thus it is an odd permutation.

(e) Why is the set $\{1, 2, 3, 4, 5\}$ under multiplication mod 6 NOT a group.

There are many reasons. One is that $2 \cdot 3 = 0$ and 0 is not in the set.

[10pts] 2. Let $x \in G$ (where the binary operation is multiplication) such that $|x| = 20$. List two elements of $\langle x \rangle$ that have order 5.

Clearly x^4 has order 5 and is in $\langle x \rangle$. We need another power of x that has order 5. If k is relatively prime to 20, then x^{k4} will also have order 5. Thus x^{12} works.

[10pts] 3. Determine if the group $G = U(12)$ is cyclic. (Hint: First list all the elements of $U(12)$.)

First note that $U(12) = \{1, 5, 7, 11\}$. We run through each element in the group to see if it generates the whole group.

$5^2 = 25 = 1 \bmod 12$; so 5 does not generate.

$7^2 = 49 = 1 \bmod 12$; so 7 does not generate.

Finally $11^2 = 121 = 120 + 1 = 1 \bmod 12$; so 11 does not generate. Hence $U(12)$ isnot cyclic.

[10pts] 4. What are the possible orders of elements of S_5?

Clearly the identity has order 1. Moreover there are cycles of length 2, 3, 4, and 5 e.g., (1 2), (1 2 3), (1 2 3 4) and (1 2 3 4 5) respectively. So there are elements in S_5 of order 2, 3, 4 and 5. Finally note there is a permutation that is the product of two disjoint cycles, one of length 2 and one of length 3. Example $\alpha = (1\ 2)(3\ 4\ 5)$. This element has order 6. Finally note that there are no permutations in S_5 that are the product of disjoint cycles of longer length. Thus the possible orders are 1, 2 ,3 ,4, 5, and 6.

[10pts] 5. Let $\beta = (123)(145)$. Write β^{92} as a product of disjoint cylces. (Hint: What is the order of β?)

First write β as a product of disjoint cycles: $\beta = (1\ 4\ 5\ 2\ 3)$. Thus $|\beta| = 5$. Hence $\beta^{92} = \beta^{90}\beta^2$. But $\beta^{90} = (\beta^5)^{18} = e^{18} = e$. Thus

$$\beta^{92} = \beta^{90}\beta^2 = \beta^2$$

Then $\beta^2 = (1\ 4\ 5\ 2\ 3)(1\ 4\ 5\ 2\ 3) = (1\ 5\ 3\ 4\ 2)$. Done.

[10pts] 6. Let $G = GL(2, R)$ and let $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \text{ and } b \text{ are nonzero integers} \right\}$. Prove or disprove that H is a subgroup of G (remember: the binary operation on G is matrix multiplication).

It is easy to check that H has an identity element and is closed under products. However it does not have inverses:

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \left\{ \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} . \right.\right.$$

But by definition, H only contains matrices with integer entries and $1/a$ is not an integer. Done.

[10pts] 7. Let $a \in G$, where $|a| = n$. Let k be a positive integer such that $a^k = e$, the identity element of G. Prove that n divides k.

We have to use the division algorithm. Write

$$k = qn + r; \text{ where } 0 \leq r < n.$$

To show that $k|n$ we must show that r above is equal to 0. Now we have

$$e = a^k = a^{qn+r} = a^{qn}a^r = (a^n)^q a^r = e^q a^r = a^r.$$

Thus $a^r = e$. But by definition n is the smallest positive integer such that $a^n = e$. Since $r < n$, we must have $r = 0$ - done.