(1) (10.16) Prove that there is no homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4 \oplus \mathbb{Z}_4$

Proof. First observe that $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = 8 \cdot 4 = 16 = 4 \cdot 4 = |\mathbb{Z}_4 \oplus \mathbb{Z}_4|$, i.e., the two groups have the same order. Hence any onto map is also one-to-one. In particular it would be an isomorphism. Thus it suffices to show that the two groups are not isomorphic. However the first group has an element of order 8 (e.g. $(1,0)$). However the largest order of any element of the second group is clearly 4, since given any $(a,b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$, each coordinate has order either 1,2 or 4.

(2) (10.20) How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_8$? How many are there to $\mathbb{Z}_8$?

Proof. If φ were an onto map, then there would be an element $g \in \mathbb{Z}_{20}$, such that $\varphi(g) = 1$. But this would mean that 8 divides $|g|$ which in turn, by Lagrange, implies that 8 divides 20, which is nonsense. Thus there are no onto maps.

To have a map from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$, we would have to first find the common divisors of 20 and 8, which are 1, 2, and 4. The only elements of $\mathbb{Z}_8$ with those orders are 0, 2, 4, and 6. To be precise $|0| = 1, |2| = 4, |4| = 2$ and $|6| = 4$ (the other elements of $\mathbb{Z}_8$ have order 8). Thus the only maps are
$\varphi : 1 \mapsto 0$,
$\varphi : 1 \mapsto 2$,
$\varphi : 1 \mapsto 4$,
$\varphi : 1 \mapsto 6$.

(3) (7.30) Suppose that $\varphi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism such that $\varphi(7) = 6$. Determine $\varphi(x)$

Proof. We have to determine $\varphi(1)$, for suppose $\varphi(1) = k$, then $\varphi(x) = kx$. To have a map $\mathbb{Z}_{50} \to \mathbb{Z}_{15}$, we need to find an element of $\mathbb{Z}_{15}$ that has order a common divisor of 50 and 15. The gcd of these two numbers is 5. So we will look for an element in $\mathbb{Z}_{15}$ of order 5. These are the elements 3, 6, 9, and 12. We try each one turn to see which one has $7 \mapsto 6$. If $\varphi(1) = 3$, then $\varphi(7) = 7\varphi(1) = 7 \cdot 3 = 21 \equiv 6 \mod 15$. Hence the first try works. Thus $\varphi(x) = 3x$ is the solution.