(1) (8.20) Find a subgroup of $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ that is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_4$.

Proof: The quickest way is to note that $\mathbb{Z}_9 \oplus \mathbb{Z}_4$ is cyclic of order 36, since 9 and 4 are relatively prime. Note that 3 as an element of $\mathbb{Z}_{12}$ has order 4 and that 2 as an element of $\mathbb{Z}_{18}$ has order 9. Thus $(2, 3)$ as an element of $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ has order $4 \cdot 9 = 36$. Hence the subgroup $H := \langle (3, 2) \rangle$ is cyclic of order 36. Hence $H \cong \mathbb{Z}_9 \oplus \mathbb{Z}_4$.

(Note: I will not require a great deal of proof for this problem.)

(2) (8.26) The group $S_3 \oplus Z_2$ is isomorphic to one of the following groups: $Z_{12}, Z_6 \oplus Z_2, A_4, D_6$. Determine which one.

Proof: We do this by a process of elimination. Clearly $G = S_3 \oplus Z_2$ is not isomorphic to either Z_{12} or $Z_6 \oplus Z_2$, since G is nonabelian, while the latter two groups are abelian (being the direct product of abelian groups.)

Next, note that $(123) \in S_3$ has order 3, while Z_2 has an element of order 2. Thus $((123), 1) \in G$ has order $\text{lcm}(3, 2) = 6$. We show that $A_4 \subset S_4$ has no element of order 6. By looking at products of disjoint cycles, we see that S_4 can have elements of order 1, 2, 3, or 4 (and this last is by taking a 4-cycle, which is not in A_4 in any case). Thus A_4 cannot have an element of order 6. Therefore, G is not isomorphic to A_4. So by elimination, G is isomorphic to D_6. (Note that we have not actually proven that the two groups are isomorphic, we are just taking the books word for it.)

(3) (9.14) What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?

Proof: We use brute force. We check to see what is the smallest multiple of 14 that lies in $H = \langle 8 \rangle$. Then

$14 \notin H$, so $14 + H \neq H$.

$2(14 + H) = 28 + H = 4 + H \neq H$.

$3(14 + H) = 42 + H = 18 + H \neq H$

$4(14) + H = 56 + H = 8 + H = H$. Thus the order of $14 + H$ in $\mathbb{Z}_{24}/\langle 8 \rangle$ is 4.

(4) (9.24) The group $G = \mathbb{Z}_4 \oplus \mathbb{Z}_{12}/\langle (2, 2) \rangle$ is isomorphic to one of $\mathbb{Z}_{12}, \mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Which one is it?

Proof: Note that $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ is abelian, so any subgroup is normal. Also note that $|\mathbb{Z}_4 \oplus \mathbb{Z}_{12}| = 48$ and $\langle (2, 2) \rangle = \{(2, 2), (0, 4), (2, 6), (0, 8), (2, 10), (0, 0)\}$. Thus the factor group G has order 8. We first claim that G has no element of order 8. One way to see this is by brute force, namely check the order of each of the 8 elements of the factor group. A quicker way is to note that the order of an element in the factor group must divide the order of the element it comes from in the original group $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$. In particular it would mean that $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$ must have an element of order divisible by 8. However, the elements of this group have order equal to the LCM of divisors of 4 and divisors of 12. None of these numbers are divisible by 8. Hence the claim is proved and so G cannot be isomorphic to $\mathbb{Z}_8$.

We next claim that the coset $(1, 0) + \langle (2, 2) \rangle$ in G has order at least 4. To see this, note that $2(1, 0) = (2, 0) \notin \langle (2, 2) \rangle$. Hence the order of $(1, 0) + \langle (2, 2) \rangle$ is at least 4 (since it cannot have order 3 by Lagrange). Thus by elimination, G is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.