Grade: ____________ KEY

[40pts] 1. The following questions are short answer and only require a brief explanation.

(a) If $\text{char}(A) = (1-\lambda)(2-\lambda)^3(5-\lambda)$, find all the eigenvalues of A.

$\lambda = 1, 2, 5$

(b) Let A and B be $n \times n$ matrices. Define what it means for A and B to be similar.

There exists an invertible matrix P such that $\boxed{A = PBP^{-1}}$

(c) Find the characteristic polynomial of $A = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$= (-1-\lambda)(1-\lambda) - 4 \to -1 + \lambda - \lambda + \lambda^2 - 4$

$= \lambda^2 - 5$

(d) Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be a basis for $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that b_1 and b_2 are eigenvectors belonging to 2 and b_3 is an eigenvector belonging to -3. Find the matrix of T with respect to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(e) Let $A = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}$. Then $A = PDP^{-1}$. Find A^6.

$A^6 = PD^6P^{-1}$ $\quad P^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} +2/3 & -1/3 \\ -1/3 & -1/3 \end{pmatrix}$

$$A^6 = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & -1/3 \end{pmatrix}$$

[10] 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, such that the matrix of T with respect to the $\mathcal{B} = \{b_1, b_2, b_3\}$ basis is

$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & -3 & 3 \end{pmatrix}$. What is $[T(b_1 - 2b_2 + b_3)]_{\mathcal{B}}$?

$$A\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & -3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+0 \\ 1-2+2 \\ 0+6+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix}$$

[10] 3. Let $u = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$. Compute $u \cdot w$ and $\|u\|^2$.

$u \cdot w = -2 + 4 + 15 = 17$

$\|u\|^2 = 4 + 4 + 9 = 17$

[8] 4. Find the distance between $u = \begin{pmatrix} 10 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

$$\|u-v\| = \left\| \begin{pmatrix} 11 \\ -2 \end{pmatrix} \right\| = \sqrt{11^2 + 2^2}$$

$$= \sqrt{121+4} = \sqrt{125}$$

$$\boxed{= 5\sqrt{5}}$$

[10] 5. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(f(x)) = f(1) + f'(2)x + f''(1)x^2.$$

Find the matrix of T with respect to the standard basis $\mathcal{B} = \{1, x, x^2\}$.

$T(1) = f(1) + f'(2)x + f''(1)x^2$

$f(1) = 1, \quad f' = 0 \quad f'' = 0$

$T(1) = 1 + 0 + 0$

$T(x) = 1 + 1x + 0x^2$

$T(x^2) = 1 + 4x + 2x^2$

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

6. The matrix $A = \begin{pmatrix} 2 & \cancel{-1} & \cancel{0} \\ 1 & 1 & \cancel{2} \\ \cancel{0} & \cancel{-3} & 3 \end{pmatrix}$ has eigenvalues 2 and 3.

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

[10] (a) Find a basis of the eigenspace for each eigenvalue.

$$A-2I = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} x_1 = -x_2 \\ x_3 = 0 \end{array}$$

Basis is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$A-3I = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} x_1 = -2x_3 \\ \left.\begin{array}{l} x_2 \\ x_3 \end{array}\right\} \text{free} \end{array}$$

Basis is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

[6] (b) Explain briefly why A is diagonalizable .

$\dim(E_2) = 1$ and $\dim(E_3) = 2$

so $\dim(E_2) + \dim(E_3) = 3 =$ # of columns of A

[6] (c) Find a diagonal matrix D, and a matrix P such that $A = PDP^{-1}$.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$