NAME (print): KEY

Math 203 Spring 2014—Exam 1

Instructor: Shapiro

Work carefully and neatly and remember that I cannot grade what I cannot read. You must show all relevant work in the appropriate space. You may receive no credit for a correct answer if there is insufficient supporting work. Notes, books and graphing or programable calculators are NOT ALLOWED.

[12] 1. Each question below has a short answer.

(a) Let A be a 6×4 matrix. How many free variables must A have in order for T_A to be one-to-one?

A must have zero free variables

(b) Let A be a 6×11 matrix. What must a and b be in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(x) = Ax$?

$a = 11$, $b = 6$

(c) Suppose that the matrix of the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ reduces to the matrix $\begin{pmatrix} 1 & 5 & 0 \\ 0 & 0 & h-2 \end{pmatrix}$. What does h have to be so that T is onto?

$h - 2 \neq 0$ or $h \neq 2$. This will ensure that every row has a pivot

[16] 2. Put each of the following augmented matrices into reduced echelon form and then describe the solution set.

(a) $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 2 & 8 \end{array}\right)$ -1

$\left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 2 & 8 \end{array}\right) \frac{1}{2}$

$\left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 8 \end{array}\right)$

$\boxed{\begin{array}{l} x_1 = -5 \\ x_2 = 4 \end{array}}$

(b) $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & 6 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right) \frac{1}{2}$

$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\boxed{\begin{array}{l} x_1 = -3x_3 + 2 \\ x_2 = -3x_3 - 4 \\ x_3 = \text{free} \end{array}}$

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[18pts] 3. Consider the following system of equations
$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 + x_4 &= 0 \\ -x_2 + 6x_3 - 5x_4 &= 0 \end{aligned}$$

(a) Find all solutions to the above system and write them in parametric vector form.

$$-2\left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & -1 & 6 & -5 & 0 \end{array}\right)$$

$$-1\left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & 6 & -5 & 0 \\ 0 & -1 & 6 & -5 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & 6 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) 1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & -2 & 0 \\ 0 & -1 & 6 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & -2 & 0 \\ 0 & 1 & -6 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

free

$x_1 = -4x_3 + 2x_4$
$x_2 = 6x_3 - 5x_4$
x_3 free
x_4 free

$$\boxed{x_3 \begin{bmatrix} -4 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}}$$

(b) Check that $v = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ is a solution to $Ax = \begin{pmatrix} 0 \\ 7 \\ 7 \end{pmatrix}$, where A is the coefficient matrix of the above system. Then use part (a) to write all solutions in parametric vector form to the matrix equation.

$$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 2 & 1 & 2 & 1 \\ 0 & -1 & 6 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0-4+3 \\ 2+0+4+1 \\ 0+0+12-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 7 \end{pmatrix} \checkmark$$

Check

All solutions of the form

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + x_3 \begin{bmatrix} -4 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

[12pts] 4. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation given by such that

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a+3b-c \\ 2c-4b \end{pmatrix}$$

(a) What is $T\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$?

$$T\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(3)+3(-2)-2 \\ 2(2)-4(-2) \end{pmatrix} = \begin{pmatrix} -2 \\ 12 \end{pmatrix}$$

(b) Find the standard matrix of T.

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -4 & 2 \end{pmatrix}$$

[16] 5. Determine what h has to be in each of the following sets of vectors so that the set spans $\mathbb{R}^3$. If no h can exist, explain why.

(a) $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ h \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ h \\ 3 \end{pmatrix}$

$$\begin{array}{c} \\ -3 \\ -6 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 6 & 3 & h \end{pmatrix}$$

$$\begin{array}{c} \\ \\ -3 \end{array}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & h \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & h-3 \end{pmatrix}$$

$h \neq 3$

$$\begin{pmatrix} -2 & 1 \\ 6 & h \\ 2 & 3 \end{pmatrix}$$

No h can exist, for in order to span $\mathbb{R}^3$, there needs to be a pivot in each row

6. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$

[10pts] (a) Determine if $\mathbf{y}$ is in the image of T. If it is, find an x such that $T(x) = y$.

$$\begin{array}{c} 1 \\ 2 \end{array} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & -1 & -5 \\ -1 & -3 & 0 \end{array}\right) \qquad 2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) -1 \qquad \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = 3$

$x_2 = -1$

y is in the image

$T\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$

[6pts] (b) Is the map T one-to-one? Why or why not?

T is one-to-one since there are no free variables

[10pts] 7. Determine if the set $A = \left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} \right\}$ is linearly independent.

$$\begin{array}{c} -5 \\ -3 \end{array} \begin{pmatrix} 1 & 2 & -2 \\ 3 & 2 & 0 \\ 5 & 4 & -4 \end{pmatrix} \qquad -1/4 \begin{pmatrix} 1 & 2 & -2 \\ 0 & -4 & 6 \\ 0 & -6 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -3/2 \\ 0 & -6 & 6 \end{pmatrix} 6 \qquad \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -3/2 \\ 0 & 0 & -3 \end{pmatrix}$$

A pivot in each column, so no free variables

YES it is independent