1.2 ROW REDUCTION, ECHELON FORMS

Echelon form (*or* **row echelon form**):

1. *All nonzero rows are above any row of zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zero.*

EXAMPLE 1 Echelon form:

$$\text{(i)}\quad \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(ii)}\quad \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{(iii)}\quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

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Reduced echelon form Add the conditions:

4. *The leading entry in each nonzero row is 1.*
5. *Each leading 1 is the only nonzero entry in its column.*

EXAMPLE 1 (continued)

Reduced echelon form:

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

THEOREM 1. UNIQUENESS OF THE REDUCED ECHELON FORM

Each matrix is row-equivalent to one and only one reduced echelon matrix.

Some terms used throughout the text:

pivot position:

a position of a leading entry in an echelon form of the matrix.

pivot:

a nonzero number that either is used in a pivot position to create 0’s or is changed into a leading 1, which in turn is used to create 0’s.

pivot column:

a column that contains a pivot position

See the Glossary at the back of the text.

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EXAMPLE 2 Row reduce to echelon form, and locate the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

$$\begin{bmatrix} \overset{\text{Pivot}}{1 \leftarrow} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ \underset{\uparrow\ \text{Pivot column}}{0} & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \overset{\text{Pivot position}}{2 \leftarrow} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & \underset{\uparrow\ \text{New pivot column}}{-3} & -6 & 4 & 9 \end{bmatrix} \qquad \text{Possible pivots: } 2, 5, -3$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & \overset{\text{Pivot}}{-5 \leftarrow} & 0 \\ \underset{\uparrow}{0} & \underset{\uparrow}{0} & 0 & \underset{\uparrow}{0} & 0 \end{bmatrix}$$

Pivot columns

Original matrix:
$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ \underset{\uparrow}{1} & \underset{\uparrow}{4} & 5 & \underset{\uparrow}{-9} & -7 \end{bmatrix}$$

Pivot columns

EXAMPLE 3 Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ \underset{\substack{\uparrow \\ \text{Pivot column}}}{3} & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Interchange rows 1 and 3.

$$\begin{bmatrix} 3\overset{\text{Pivot}}{\leftarrow} & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Add −1 times row 1 to row 2.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is echelon form.

Pivot

Final Step to Create the Reduced Echelon Form::

Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Here

Then here

$$\approx \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

basic variable:

any variable that corresponds to a pivot column in the augmented matrix of a system.

free variables:

all nonbasic variables.

EXAMPLE 4′

$$\begin{array}{cc}
\left[\begin{array}{rrrrrr} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] &
\begin{array}{lcrcr} x_1 + 6x_2 & & + 3x_4 & & = 0 \\ & & x_3 - 8x_4 & & = 5 \\ & & & & x_5 = 7 \end{array} \\
\begin{array}{cccccc} \uparrow & \quad & \uparrow & \quad & \uparrow & \quad \end{array} &
\begin{array}{ccccc} \uparrow & \qquad\qquad & \uparrow & \qquad\qquad & \uparrow \end{array}
\end{array}$$

pivot columns: 1, 3, 5 basic variables: $x_1,\ x_3,\ x_5$

free variables: x_2 and x_4

Final Step in Solving a Consistent Linear System

After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

> *Solve each equation for the basic variables in terms of the free variables (if any) in the equation.*

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EXAMPLE

$$\begin{aligned} x_1 + 6x_2 \qquad\quad + 3x_4 \qquad &= 0 \\ x_3 - 8x_4 \qquad &= 5 \\ x_5 &= 7 \end{aligned} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.)

BACK-SUBSTITUTION

The following system is in echelon form but is *not* in reduced echelon form.

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\ x_4 - x_5 &= 4 \end{aligned}$$

The backward phase of row reduction, to reduced echelon form, is equivalent to back-substitution. Use only the **reduced** echelon form to solve a system.

A Geometric Description of Span{**u**, **v**}

Take **u** and **v** in $\mathbb{R}^3$, with **v** not a multiple of **u**.

$$\begin{aligned}\text{Span}\{\mathbf{u}, \mathbf{v}\} &= \text{plane containing } \mathbf{u}, \mathbf{v}, \text{ and the origin } \mathbf{0}. \\ &= \text{the plane in } \mathbb{R}^3 \textbf{ spanned} \text{ by } \mathbf{u} \text{ and } \mathbf{v}.\end{aligned}$$

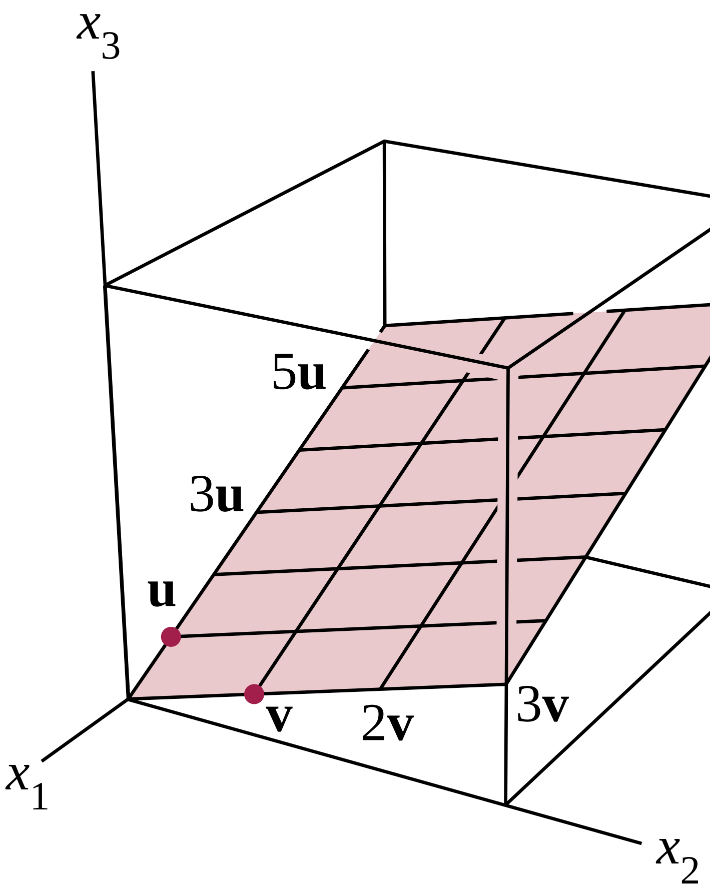

FIGURE 11 Span{**u**, **v**} as a plane through the origin.

Visualize Span{**u**, **v**} as a plane through the origin, whenever **u** and **v** are in $\mathbb{R}^n$ and **v** is not a multiple of **u**.

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