Exam 2 Solutions

1. a) Let us show that u_t is <u>not</u> frame indifferent.

$$X^* = X + b(t) \Rightarrow U^* = U + b(t) \Rightarrow$$

$$u^* = u + b(t) \Rightarrow u^*(x^*, t^*) = u(x,t) + b(t)$$

Differentiate the last equation with respect to t: $u^*_{x^*} \cdot x^*_t + u^*_{t^*} \cdot t^*_t = u_x \cdot x_t + u_t + b'(t)$

$$\Rightarrow u^*_{x^*}(\underbrace{x_t + b'_t}_{\text{in general}}) + \boxed{u^*_{t^*}} = u_x \cdot x_t + \boxed{u_t} + b'(t)$$

$b'(t) \neq 0$

$\Rightarrow u^*_t \neq u_t \Rightarrow \tau = \tau(u_t)$ cannot satisfy the PMFI.

b) Similarly, $V^* = V + b'(t) \Rightarrow$

$$v^* = v + b'(t) \Rightarrow v^*(x^*, t^*) = v(x,t) + b'(t)$$

$$\Rightarrow v^*_{x^*} \cdot x^*_t + v^*_{t^*} \cdot t^*_t = v_x \cdot x_t + v_t + b''(t) \Rightarrow$$

$$v^*_{x^*}(\underbrace{x_t + b'_t}_{\text{in general}}) + \boxed{v^*_{t^*}} = v_x \cdot x_t + \boxed{v_t} + b''(t) \Rightarrow$$

$v^*_{t^*} \neq v_t \Rightarrow \tau = \tau(v_t)$ cannot satisfy the PMFI.

c) $u^*(x^*, t^*) = u(x,t) + b(t)$ ← Differential w.r.t. x.

$u^*_{x^*} \cdot \underbrace{x^*_x}_{1} = u_x \Rightarrow u^*_{x^*} = u_x$ and u_x is frame indifferent

However, differentiate w.r.t. t.

⇓

$$u^*_{x^*x^*}\cdot x^*_t + u^*_{x^*t^*}\cdot t^*_t = u_{xx}\cdot x_t + u_{xt} \Rightarrow$$ 2

$$u^*_{x^*x^*}(x_t+b'_t) + \boxed{u^*_{x^*t^*}} = u_{xx}\cdot x_t + \boxed{u_{xt}} \Rightarrow$$

$b'_t \neq 0$

$u^*_{x^*t^*} \neq u_{xt}$ in general $\Rightarrow$

u_{xt} is <u>not</u> frame indifferent $\Rightarrow$

$\Rightarrow \quad \tau = \tau(u_{xt})$ cannot satisfy the PMFI.

$\boxed{2}$ $\quad \varepsilon_a = u_x - 0.5u_x^2 \;;\; \Psi = \Psi(\varepsilon_a),\; \tau = E\varepsilon_a$

From (6.77) of the book, we have

$$\tau = \rho(1-u_x)\Psi'(u_x) = \rho(1-u_x)\Psi'_{\varepsilon_a}\cdot\frac{\partial\varepsilon_a}{\partial u_x} =$$

$$= \rho(1-u_x)^2\Psi'_{\varepsilon_a} \Rightarrow$$

$$\tau = E\varepsilon_a = \rho(1-u_x)^2\Psi'_{\varepsilon_a} \quad (*)$$

$$0.5u_x^2 - u_x + \varepsilon_a = 0 \Rightarrow 0.5(u_x^2 - 2u_x + 1 - 1) + \varepsilon_a = 0$$

$$\Rightarrow 0.5((u_x-1)^2 - 1) + \varepsilon_a = 0 \Rightarrow (u_x-1)^2 - 1 = -2\varepsilon_a$$

$\Rightarrow (u_x-1)^2 = 1 - 2\varepsilon_a \Rightarrow (*)$ becomes

$$E\varepsilon_a = \rho(1-2\varepsilon_a)\Psi'_{\varepsilon_a} \Rightarrow \Psi'_{\varepsilon_a} = \frac{E}{\rho}\frac{\varepsilon_a}{1-2\varepsilon_a} =$$

$$= \frac{E}{2\rho}\frac{2\varepsilon_a}{1-2\varepsilon_a} = \frac{E}{2\rho}\frac{2\varepsilon_a - 1 + 1}{1-2\varepsilon_a} = \frac{E}{2\rho}\left(-1 + \frac{1}{1-2\varepsilon_a}\right)$$

$\Downarrow$

$$\psi(\varepsilon_a) = \frac{E}{2\rho}\left(-\varepsilon_a - \frac{1}{2}\ln|1-2\varepsilon_a|\right) + C$$ 3

We will requile that $\psi(0) = 0 \Rightarrow C = 0$

Also since $1 - 2\varepsilon_a > 0 \Rightarrow$

$$\psi(\varepsilon_a) = \frac{E}{2\rho}\left(-\varepsilon_a - \ln\sqrt{1-2\varepsilon_a}\right)$$

$\boxed{3}$ $$\frac{\partial^2 U}{\partial t^2} - 9\frac{\partial^2 U}{\partial A^2} = t, \quad U(A,t) = f(A), \quad U_t(A,t) = 0$$

For the homogeneous solution U_h:

$$\frac{\partial^2 U_h}{\partial t^2} - 3^2\frac{\partial^2 U_h}{\partial A^2} = \left(\frac{\partial}{\partial t} - 3\frac{\partial}{\partial A}\right)\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial A}\right)U_h =$$

$= \frac{\partial}{\partial r}\frac{\partial}{\partial s}U_h = 0$, we have $U_h = \bar{F}(r) + \bar{G}(s)$

with $r = -\frac{1}{6}(A-3t)$, $s = \frac{1}{6}(A+3t) \Rightarrow$

$$U_h = F(A-3t) + G(A+3t)$$

Note that the particular solution $P(A,t) = \frac{t^3}{6}$ satisfies the PDE $\Rightarrow$

A general solution is $U(A,t) = F(A-3t) + G(A+3t) + \frac{t^3}{6}$

↓

Let's use the initial conditions (4)
to find F and G

$t=0 \Rightarrow F(A)+G(A)=f(A)$

$-3F'(A)+3G'(A)=0 \Rightarrow F'(A)=G'(A) \Rightarrow$

$F(A)=G(A)+C$, so:

$$\begin{aligned} F(A)+G(A) &= f(A) \\ F(A)-G(A) &= C \end{aligned} \Rightarrow F(A)=\frac{1}{2}f(A)+\frac{C}{2} \Rightarrow G(A)=\frac{1}{2}f(A)-\frac{C}{2}$$

$$U(A,t)=\frac{1}{2}f(A-3t)+\cancel{\frac{C}{2}}+\frac{1}{2}f(A+3t)-\cancel{\frac{C}{2}}+\frac{t^3}{6}$$

$$=\frac{f(A-3t)+f(A+3t)}{2}+\frac{t^3}{6}$$

$\boxed{4}$ $y''+2y'+y=0$, $y(0)=0$, $y'(0)=1$

$L(y'')=s^2\hat{y}-y'(0)-sy(0)=s^2\hat{y}-1$, $\boxed{\hat{y}=\mathcal{L}(y)}$

$L(y')=s\hat{y}-y(0)=s\hat{y} \Rightarrow$

$\mathcal{L}(y''+2y'+y)=s^2\hat{y}-1+2s\hat{y}+\hat{y}=0 \Rightarrow$

$\hat{y}(s^2+2s+1)=1 \Rightarrow \hat{y}=\frac{1}{(s+1)^2} \Rightarrow \boxed{y=te^{-t}}$

$\boxed{5}$

$$x' = -x + 2y \qquad x(0) = 0$$
$$y' = 2x + y \qquad y(0) = -1$$

Using the notations $\mathcal{L}(x) = \hat{x}$, $\mathcal{L}(y) = \hat{y}$,

we have

$$\mathcal{L}(x') = \mathcal{L}(-x+2y)$$
$$\mathcal{L}(y') = \mathcal{L}(2x+y)$$
, or

$$s\hat{x} - x(0) = -\hat{x} + 2\hat{y}$$
$$s\hat{y} - y(0) = 2\hat{x} + \hat{y}$$
, or

$$\hat{x}(s+1) - 2\hat{y} = 0 \quad (1)$$
$$2\hat{x} + \hat{y}(1-s) = +1 \quad (2)$$

To eliminate $\hat{y}$, multiply (1) by $(1-s)$, (2) by 2 and add: $\hat{x}\left(-(s+1)(s-1)+4\right) = 2 \Rightarrow \hat{x} = \frac{2}{}$

$$\hat{x}(-s^2+1+4) = 2 \Rightarrow \hat{x} = -\frac{2}{s^2-5} = \frac{-2}{(s-\sqrt{5})(s+\sqrt{5})}$$

$$\Rightarrow \boxed{x = -\frac{2}{2\sqrt{5}}\left(e^{\sqrt{5}t} - e^{-\sqrt{5}t}\right)}$$

To eliminate $\hat{x}$, multiply (2) by $(s+1)$ and subtract from (1) multiplied by 2

$$\hat{y}(1-s)(s+1) + 4\hat{y} = s+1 \Rightarrow \hat{y}(5-s^2) = s+1$$

$$\hat{y} = -\frac{s+1}{(s-\sqrt{5})(s+\sqrt{5})} \Rightarrow \boxed{y = -\frac{1}{2\sqrt{5}}\left((\sqrt{5}-1)e^{-\sqrt{5}t} + (\sqrt{5}+1)e^{\sqrt{5}t}\right)}$$