1. **(18 points)** List the elements of the set

$$A = \{(a, b) \in \mathbf{N} \times \mathbf{N} |\, a \geq b,\ a \leq 3\}.$$

a) What is the cardinality of A (the number of elements in the set)?

b) Is A a binary relation on $B = \{1, 2, 3\}$?

c) If yes, is it reflexive, symmetric, antisymmetric, transitive?

d) Is it an equivalence relation?

e) Is it a partial order?

Justify all your answers.

Solution.

$$A = \{(1, 1), (2, 1), (3, 1), (2, 2), (3, 2), (3, 3)\}.$$

a) The cardinality is 6.

b) Yes. A is a binary relation because it is is a subset of the cartesian product $B \times B$.

c) It is reflexive because $(1, 1)$, $(2, 2)$, $(3, 3) \in A$.

It is not symmetric because, for example, $(1, 2) \in A$, but $(2, 1) \notin A$.

It is antisymmetric because only $(1, 1)$, $(2, 2)$, and $(3, 3)$ are such that simultaneously $(a, b) \in A$ and $(b, a) \in A$.

It is transitive because whenever $(a, b) \in A$ and $(a, c) \in A$, (a, c) is also $\in A$.)

d) It is not an equivalence relation because it is not symmetric.

e) It is a partial order because it is reflexive, antisymmetric and transitive.

2. **(18 points)**

p	q	r	$(p$	$\rightarrow$	$(q$	$\rightarrow$	$r))$	$\longrightarrow$	$((p$	$\wedge$	$q)$	$\vee$	$r)$
T	T	T		T		T		**T**		T		T	—
T	T	F		F		F		**T**		T		T	—
T	F	T		T		T		**T**		F		T	—
T	F	F		T		T		**F**		F		F	—
F	T	T		T		T		**T**		F		T	—
F	T	F		T		F		**F**		F		F	—
F	F	T		T		T		**T**		F		T	—
F	F	F		T		T		**F**		F		F	—

3. **(7 points)** Let $n = ab$ be the product of positive integers a and b. Prove that either $a \geq \sqrt{n}$ or $b \geq \sqrt{n}$.

Solution. Suppose that both $a < \sqrt{n}$ and $b < \sqrt{n}$. Then $ab < \sqrt{n}\sqrt{n} = n$, which contradicts to $ab = n$. Therefore either $a \geq \sqrt{n}$ or $b \geq \sqrt{n}$.

4. **(14 points)** Prove that the sum of a rational and an irrational numbers is an irrational number.

   Solution. Assume a and b be the rational and irrational numbers respectively. Suppose that $c = a + b$ is a rational number. So we have $a = m/n$, $c = l/k$, where m, n, l, k are integers. So we have $l/k = m/n + b$, or $b = l/k - m/n = (ln - km)/(kn)$. $ln - km$ is integer, kn is ineger, therefore b is rational, which contradicts our asumption. Therefore c cannot be a rational number.

   Is a sum of two irrational numbers an irrational number? No. Example: $-\pi + \pi = 0$.

5. **(18 points)** Determine the validity of the following argument:

```
If I work hard, then I earn lots of money.
If I earn lots of money, then I pay high taxes.
---------------------------------------------------
If I do not work hard, then I do not pay high taxes.
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   Solution. Let a, b and c be the arguments:

   a : I work hard
   b : I earn lots of money
   c : I pay high taxes

   Then the given argument is

$$a \to b$$

$$b \to c$$

$$----$$

$$\neg a \to \neg c$$

   Solution. Constructing the truth table you can see that the argument is invalid e.g. if a is FALSE, b is TRUE and c is TRUE. Then $a \to b$ is TRUE, $b \to c$ is TRUE but $\neg a \to \neg c$ is FALSE. In other words, if I do NOT work hard but still make lots of money somehow, I will have to pay high taxes.

6. **(7 points)** Show that there exists no largest negative rational number.

   Solution. Suppose that there exists the largest negative rational number a. Then another rational negative number $b = 0.5a$ is actually greater than a. Therefore we come to the contradiction and therefore there exists no largest negative rational number.

7. **(18 points)** Prove without the use of Venn diagram that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution1. Let $x \in A$, $x \in B$, and $x \in C$, are logical statement a, b, and c respectively. So we have

$$x \in A \cup (B \cap C) \Leftrightarrow x \in A \,\text{or}\, x \in B \cap C \Leftrightarrow x \in A \,\text{or both}\, x \in B \,\text{and}\, x \in C \Leftrightarrow$$

$$a \vee (b \wedge c) \Leftrightarrow \{DeMorgan's\, law\} \Leftrightarrow (a \vee b) \wedge (a \vee c) \Leftrightarrow$$

$$\text{both}\,(x \in A \,\text{or}\, x \in B)\,\text{and}\,(x \in A \,\text{or}\, x \in C) \Leftrightarrow$$

$$\text{both}\,(x \in A \cup B)\,\text{and}\,(x \in A \cup C) \Leftrightarrow x \in (A \cup B) \cap (A \cup C).$$

Therefore the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ consist of same elements.

Solution2. Let $x \in A \cup (B \cap C)$. Then two cases are possible. Case 1: $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$. Case 2: $x \in (B \cap C)$. Then $x \in B$ and $x \in C$. Then $x \in A \cup B$ and $x \in A \cup C$. Then $x \in (A \cup B) \cap (A \cup C)$. Both cases lead to $x \in (A \cup B) \cap (A \cup C)$. Therefore $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$. Case 1: $x \in A$. Then $x \in A \cup (B \cap C)$. Case 2: $x \notin A$. Then since $x \in (A \cup B)$ and $x \in (A \cup C)$ we have $x \in B$ and $x \in C$. Then $x \in B \cap C$. Then $x \in A \cup (B \cap C)$. Both cases lead to $x \in A \cup (B \cap C)$. Therefore $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Since $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$, we have $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.