On the Sylvester Denumerants for General Restricted Partitions

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Abstract

Let n be a nonnegative integer, and let $\tilde{a} = (a_1, \ldots, a_k)$ be a k-tuple of positive integers. The term *denumerant*, introduced by Sylvester, denotes the number $D(n; \tilde{a})$ of ways one can partition the number n into parts $a_1, \ldots, a_k$. In this article we use direct combinatorial methods to find concrete and simply expressible upper and lower bounds for $D(n; \tilde{a})$ in terms of n, k and general $\tilde{a}$, both bounds having the known asymptotic value of $D(n; \tilde{a})$ for large n. Finally we derive additional properties of $D(n; \tilde{a})$ that hold for infinitely many n.

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1 Introduction

Partitions of a nonnegative integer n have been studied extensively. Some types of partitions have already been studied intensively, while other types are still being studied, although from a different perspective than was done in the late nineteenth century. The use of generating functions has been proved to be invaluable in the study of partitions, but knowing the generating function for a certain combinatorial number sequence does not always yield accessible information for the working mathematician and computer scientist.

The purpose of this article is to demonstrate that in certain situations using direct combinatorial methods and elementary number theory can provide more concrete information than by considering the corresponding generating function.

We will be considering general restricted partitions of a nonnegative integer n, defined in Definition 1.1. By a *restricted partition* of n into k

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parts, we mean a decomposition $n = y_1 + \cdots + y_k$, where $y_1 \geq \cdots \geq y_k \geq 1$ are integers. The number of such decompositions $y_1, \ldots, y_k$ will be denoted by $p_k(n)$. Note that if we let $y_i' = y_i - 1$ for each $i \in \{1, \ldots, k\}$, then we see that $p_k(n+k)$ is the number of decompositions $n = y_1' + \cdots + y_k'$, where $y_1' \geq \cdots \geq y_k' \geq 0$.

Consider now the equation $x_1 + 2x_2 + \cdots + kx_k = n$. Denote the number of solutions in nonnegative integers $x_1, \ldots, x_k$ by $d_k(n)$. If we let $y_i' = x_i + x_{i+1} + \cdots + x_k$ for each $i \in \{1, \ldots, k\}$, then this change of variables shows that $d_k(n) = p_k(n+k)$, or equivalently $p_k(n) = d_k(n-k)$. Since now the generating function for the $d_k(n)$'s where $n \geq 0$, is given by

$$\sum_{n \geq 0} d_k(n) X^n = \frac{1}{(1-X)(1-X^2)\cdots(1-X^k)},$$

the number of restricted partitions, $p_k(n)$, can be read from the expansion of this generating function also. Hence, we can view restricted partitions as the number of nonnegative integer solutions to a certain linear equation.

Consider a natural generalization of the above, the partition of n into specified parts. As stated in [1, p. 117], the following definition is due to J. J. Sylvester.

Definition 1.1 *Let n be a nonnegative integer and $\tilde{a} = (a_1, \ldots, a_k)$ be a k-tuple of positive integers. A partition of n into positive parts $a_1, \ldots, a_k$ is a decomposition*

$$n = a_1 x_1 + \cdots + a_k x_k, \tag{1}$$

where $x_1, \ldots, x_k$ are nonnegative integers. The number of such solutions in $x_1, \ldots, x_k$ to (1) is called denumerant *and is denoted by $D(n; \tilde{a})$ or $D(n; a_1, \ldots, a_k)$.*

Remarks: (i) The term "denumerant" is also due to Sylvester, who seemed to have been quite inventive when it came to new words and phrases. (ii) The problem of finding all solutions to (1) is sometimes called *the money change problem*, since it can be phrased as: "Find all ways one can change n dollars into coins or bills of size $a_1, \ldots, a_k$," [2], [3, p. 327].

If $D_{\tilde{a}}(X)$ denotes the generating function for the $D(n; \tilde{a})$'s where $n \geq 0$ for a fixed k-tuple $\tilde{a}$, then

$$D_{\tilde{a}}(X) = \sum_{n \geq 0} D(n; \tilde{a}) X^n = \frac{1}{(1-X^{a_1})(1-X^{a_2})\cdots(1-X^{a_k})},$$

and hence we have in particular that $D(n; 1, \ldots, k) = d_k(n) = p_k(n+k)$. It is precisely this generating function $D_{\tilde{a}}(X)$ that is quite hard to expand for a *general* k-tuple $\tilde{a}$.

2 Some History

Since denumerants count general kinds of restricted partitions of n, their history is like that of partitions in general, and is split into two eras, the old school and the new school.

The old school started around 1850 and lasted about one century until 1950 or so. Some scholars have pinpointed the end of this era as 1917 [4], [5], by the announcement of “Une Formule Asymptotique pout le Nombre des Partitions de n,” Comptes Rendus, in January 2nd, 1917, and later the appearance of the full paper [6] on the circle method for asymptotics. This apparently pushed the Cayley-Sylvester ideas to the background. Still some papers, with the old school flavor, were published after 1917, but to a much less extent than before. The old school includes work of many pioneers such as A. Cayley, J. J. Sylvester, P. A. MacMahon, A. DeMorgan, G. H. Hardy, E. M. Wright, S. Ramanujan, and E. T. Bell to name a few. The results from this era can be characterized by powerful use of generating functions. This was the era of exactness and hardly any estimations nor approximations were made.

The new school includes estimations and some special cases which were left over by the old school, some “neat tricks” for an exact solution for a given k-tuple $\tilde{a}$. Special cases have been considered, when a general exact solution is hard to obtain. In [7, p. 113] the case where $k \leq 4$ is considered, and in [2] some algorithmic approximations for the case $k = 3$ are considered, where the a_i's are further assumed to be pairwise relatively prime. Some identities and methods of computing are considered in [8] and [9]. In [10] an expression for the generating function $D_{\tilde{a}}(X)$ is given as $P(\frac{1}{1-X}) + R(X)$, where $P(X)$ is a polynomial and $R(X)$ is a rational function with a denominator not divisible by $1 - X$. There an explicit formula for $P(X)$ is given in terms of Bernoulli numbers. In [11] an upper bound for $D(n; \tilde{a})$ is given in terms of n, k and $\tilde{a}$, which is similar to what will obtained in Section 3. We will improve this upper bound, provide a lower bound and some further results that hold for infinitely many n, that are evenly distributed among the nonnegative integers.

For each *specific* k-tuple $\tilde{a}$ one can compute $D(n; \tilde{a})$ exactly as a function of n by considering the partial fraction decomposition of the generating function $D_{\tilde{a}}(X)$, and then using the extended binomial theorem on terms of the form $1/(1-\xi X)^i$ to write $D(n; \tilde{a})$ as a linear combination of binomial coefficients. As an example of this method, one can derive De Morgan's result stated in [1, p. 119] that $D(n; 1, 2, 3) = d_3(n) = [(n + 3)^2/12]$, where $[x]$ here denotes the integer nearest to x. Precisely this is demonstrated in [12, p. 132]. Another nice example can be found in [3, p. 344], where full advantage is taken of the particular values of the $a_1, \ldots, a_k$ to obtain the partial fraction decomposition of the generating function $D_{\tilde{a}}(X)$ and hence an exact formula for $D(n;\tilde{a})$.

Although this partial fraction method is primarily used to compute exact values of $D(n;\tilde{a})$ for specific $\tilde{a}$, it can also be used to obtain different exact results. As an example, the following "old-school-like" theorem is due to E. T. Bell [13]:

Theorem 2.1 *Let n be a nonnegative integer and $\tilde{a}$ a k-tuple of positive integers $a_1, \ldots, a_k$. If a is the least common multiple of all the a_i, then for each $b \in \{0, 1, \ldots, a-1\}$ the denumerant $D(am+b;\tilde{a})$ is a polynomial in m of degree $k-1$.*

Bell's proof in [13] is based on fact that in the partial fraction decomposition of the generating function $D_{\tilde{a}}(X)$, every ξ in each term $1/(1-\xi X)^i$ is an m-th root of unity, which in particular implies that ξ^{am+b} is fixed for all $m \geq 0$.

The asymptotic behavior of $D(n;\tilde{a})$ can also be obtained by considering the form of the partial fraction decomposition of $D_{\tilde{a}}(X)$ if $\gcd(a_1, \ldots, a_k) = 1$: Since there is only one term $1/(1-\xi X)^i$ with $i = k$, and that term has $\xi = 1$, all other terms have $i \leq k-1$, and hence $1/(1-X)^k$ is the only contributor to the coefficient for n^{k-1} in $D(n;\tilde{a})$. From this we can obtain that

$$D(n;\tilde{a}) \sim \frac{n^{k-1}}{(k-1)!a_1 \cdots a_k}, \tag{2}$$

where $f(n) \sim g(n)$ denotes $\lim_{n\to\infty} f(n)/g(n) = 1$. To demonstrate this asymptotic value, using the mentioned partial fraction decomposition, is suggested as an exercise in [12, p. 134]. We note that if exactly $k-1$ variables of $x_1, \ldots, x_k$ are given, then the remaining one is determined by (1). Hence, it should come as no surprise that the asymptotic value given in (2) has degree at most $k-1$ in n.

For a *general* k-tuple $\tilde{a}$ it gets harder to obtain further results using the generating function $D_{\tilde{a}}(X)$. A result of Sylvester [13] states that $D(n;\tilde{a}) = A(n) + U(n)$, where $A(n)$ is a polynomial of degree $k-1$ and $U(n)$ is the "undulant" part, which contains roots of unity. (This is yet another example of a new phrase invented by Sylvester!) Here

$$A(n) = \sum_{j=0}^{k-1} \alpha_{k-1-j} \frac{n^j}{j!},$$

where α_s is the coefficient of X^s in $\prod_{j=1}^{k}(1-e^{-a_j X})^{-1}$. As we can see, it is hard to obtain any concrete values from this nice result. First of all, we do not know from this alone about the amplitude of the undulant, how large it is in terms of $\tilde{a}$. Second of all, it seems to be equally as hard to obtain the coefficients α_s as it is to directly obtain $D(n;\tilde{a})$ from the generating function $D_{\tilde{a}}(X)$.

Sometimes more accessible information can be derived by considering direct combinatorial arguments. As an example from [12, p. 133], one can use combinatorial arguments to prove a special case of (2), that for a fixed integer k we have $D(n; 1, \ldots, k) \sim \frac{n^{k-1}}{(k-1)!k!}$. The proof given provides *concrete* upper and lower bounds for $D(n; 1, \ldots, k)$, both of which have the same asymptotic value for large n. In fact, the proof yields

$$\frac{1}{k!}\binom{n+k-1}{k-1} \leq D(n; 1, \ldots, k) \leq \frac{1}{k!}\binom{n+k(k+1)/2-1}{k-1}. \qquad (3)$$

Hence, a natural question is whether similar concrete bounds can also be obtained for the *general* denumerant $D(n; \tilde{a})$. In the next section we obtain just that.

3 Concrete Bounds

In this section we will derive concrete upper and lower bounds for $D(n; \tilde{a})$ for a general k-tuple $\tilde{a}$ by using elementary number theoretic and purely combinatorial arguments.

NOTATION: Before starting we need to introduce some useful notation. Let $\tilde{v}$ always denote a k-tuple $(v_1, \ldots, v_k)$. For each $i \in \{1, \ldots, k\}$ we let $\tilde{v}_i = (v_1, \ldots, v_i)$ be the i-tuple of the first i entries of $\tilde{v}$. We let $\Pi\tilde{v} = v_1 \cdots v_k$ be the product of all the entries, and $\gcd(\tilde{v})$ denote $\gcd(v_1, \ldots, v_k)$. Hence, in particular, $\gcd(\tilde{v}_i) = \gcd(v_1, \ldots, v_i)$ and $\Pi\tilde{v}_i = v_1 \cdots v_i$ for each $i \in \{1, \ldots, k\}$. Further, $|\tilde{v}|$ will always mean the usual L^1-norm of $\tilde{v}$, that is $|v_1| + \cdots + |v_k|$. In particular when $\tilde{v}$ is a k-tuple of nonnegative integers, then $|\tilde{v}| = v_1 + \cdots + v_k$. For a finite set $\mathcal{S}$ however, $|\mathcal{S}|$ will mean the cardinality of $\mathcal{S}$. For two k-tuples $\tilde{v}$ and $\tilde{w}$ their dot product $v_1w_1 + \cdots + v_kw_k$ will be denoted by $\tilde{v} \cdot \tilde{w}$. In particular, (1) can be written as $n = \tilde{a} \cdot \tilde{x}$. If t is a real number, then $\lfloor t \rfloor$ denotes the largest integer $\leq t$, and $\lceil t \rceil$ denotes the smallest integer $\geq t$.

Turning our attention back to denumerants, we now state two basic facts, both of which are necessary in our arguments to come.

Observation 3.1 *For the two tuple* $\tilde{a} = (a_1, a_2)$ *let* $d = \gcd(a_1, a_2)$. *If there is one nonnegative solution* (x_1^*, x_2^*) *to* $a_1x_1 + a_2x_2 = n$, *then there is one with* $x_2^* \in \{0, \ldots, a_1/d - 1\}$, *and all the nonnegative solutions are determined by* $x_2 = x_2^* + ia_1/d$, *where* $i \in \{0, \ldots, \lfloor d(n - a_2x_2^*)/a_1a_2 \rfloor\}$.

REMARK: The fact that (x_1^*, x_2^*) is indeed a nonnegative solution ensures us that $d(n - a_2x_2^*)/a_1a_2 = dx_1^*/a_2 \geq 0$. Hence, the range for i in Observation 3.1 is proper.

Lemma 3.2 *Let P,q,r be nonnegative integers and $f(i)=P-qi$ a linear map with $f(0)=P\geq r-1$. If $N=\lfloor (P-r+1)/q\rfloor$ is the largest nonnegative integer with $f(N)\geq r-1$ then we have*

$$\frac{1}{q}\binom{f(0)+1}{r}\leq\sum_{i=0}^{N}\binom{f(i)}{r-1}\leq\frac{1}{q}\binom{f(-1)}{r}.$$

Proof. For the upper bound note that $\binom{f(-1)}{r}=\sum_{h=0}^{f(-1)-r}\binom{f(-1)-h-1}{r-1}$. Grouping together every q consecutive terms, from the first term $\binom{f(-1)-1}{r-1}$ we get

$$\begin{aligned}\binom{f(-1)}{r} &= \sum_{i=0}^{N}\left(\sum_{j=1}^{q}\binom{f(i)-j+q}{r-1}\right)+\sum_{h=0}^{f(N)-r}\binom{f(N)-h-1}{r-1}\\ &\geq q\sum_{i=0}^{N}\binom{f(i)}{r-1}+\binom{f(N)}{r},\end{aligned}$$

from which we obtain the upper bound.

For the lower bound we likewise have $\binom{f(0)+1}{r}=\sum_{h=0}^{f(0)-r+1}\binom{f(0)-h}{r-1}$ Here we group together every q consecutive terms starting with the initial term $\binom{f(0)}{r-1}$ as far down as we are able. Hence, we have the following inequality, where we note that $f(N)-r+1\leq q-1$ by the definition of N.

$$\begin{aligned}\binom{f(0)+1}{r} &= \sum_{i=0}^{N-1}\left(\sum_{j=0}^{q-1}\binom{f(i)-j}{r-1}\right)+\sum_{j=0}^{f(N)-r+1}\binom{f(N)-j}{r-1}\\ &\leq q\sum_{i=0}^{N}\binom{f(i)}{r-1},\end{aligned}$$

which yields the lower bound, and hence completes the proof. □

REMARK: Clearly the upper bound in Lemma 3.2 holds for *any* nonnegative integer $N\leq\lfloor (P-r+1)/q\rfloor$, but the lower bound only holds when $N=\lfloor (P-r+1)/q\rfloor$.

Our first result provides a concrete upper bound for $D(n;\tilde{a})$ for a general k-tuple $\tilde{a}$. This upper bound has the same asymptotic value as the one given in (2) when $\gcd(\tilde{a})=1$.

For a fixed k-tuple $\tilde{a}$ we define k integers $A_1,\ldots,A_k$ recursively by

$$\begin{aligned}A_1 &= 0,\\ A_i &= A_{i-1}+a_i\frac{\gcd(\tilde{a}_{i-1})}{\gcd(\tilde{a}_i)},\ \text{for } 2\leq i\leq k.\end{aligned}$$

Note that for all i we have $A_i \geq i - 1$. For the rest of this article A_i will always have the above meaning, unless otherwise stated. The following theorem is an improvement of the upper bound given in [11].

Theorem 3.3 *For a k-tuple $\tilde{a}$ of positive integers let $d = \gcd(\tilde{a})$. We then have*

$$D(n;\tilde{a}) \leq \frac{d}{\Pi\tilde{a}}\binom{n + A_k}{k-1}.$$

Proof. For $k = 1$ it is a tautology and for $k = 2$ the theorem is true by Observation 3.1.

We proceed by induction on k and assume we have the theorem for $k - 1 \geq 1$. If $d' = \gcd(\tilde{a}_{k-1})$ then $d = \gcd(d', a_k)$. If we now let $x = (a_1/d')x_1 + \cdots + (a_{k-1}/d')x_{k-1}$, then (1) becomes the two variable equation $d'x + a_k x_k = n$. If (1) has a nonnegative solution (i.e. all variables are nonnegative,) then by Observation 3.1 this corresponding two variable equation has a nonnegative solution (x, x_k) with $x_k = x_k^* \in \{0, \ldots, d'/d - 1\}$, and all the nonnegative solutions to (1) must have $x_k = x_k^* + id'/d$ for some $i \in \{0, \ldots, \lfloor d(n - a_k x_k^*)/d'a_k \rfloor\}$. Since nonnegative solutions to (1) are included among those with x_k nonnegative, we can deduce the following recursive formula.

$$D(n;\tilde{a}) = \sum_{i=0}^{N} D(n - a_k(x_k^* + id'/d); \tilde{a}_{k-1}), \tag{4}$$

where N is the largest nonnegative integer with $n \geq a_k(x_k^* + Nd'/d)$. Letting $b(i) = n + A_{k-1} - a_k(x_k^* + id'/d)$ for each i, the recursive formula (4) becomes

$$D(n;\tilde{a}) = \sum_{i=0}^{N} D(b(i) - A_{k-1}; \tilde{a}_{k-1}). \tag{5}$$

Since $A_{k-1} \geq k - 2$ we have by induction hypothesis, (5) and Lemma 3.2 that

$$D(n;\tilde{a}) \leq \frac{d'}{\Pi\tilde{a}_{k-1}}\left(\sum_{i=0}^{N}\binom{b(i)}{k-2}\right) \leq \frac{d}{\Pi\tilde{a}}\binom{b(-1)}{k-1} \leq \frac{d}{\Pi\tilde{a}}\binom{n + A_k}{k-1},$$

which completes the proof. □

Remark: We note that $(1 - X^{a_k})D_{\tilde{a}}(X) = D_{\tilde{a}_{k-1}}(X)$, and hence we get the recursive formula $D(n;\tilde{a}) - D(n - a_k;\tilde{a}) = D(n;\tilde{a}_{k-1})$. Summing up $D(n - ia_k;\tilde{a}) - D(n-(i+1)a_k;\tilde{a}) = D(n - ia_k;\tilde{a}_{k-1})$ for each $i \in \{0, 1, \ldots, \lfloor n/a_k \rfloor - 1\}$, would seem to imply an easier approach than the one given by using (4) in the above proof. However, the difficulty here is that most terms $D(n - ia_k;\tilde{a}_{k-1})$ are zero (roughly $(d' - d)/d$ of them.) Hence, the upper bound obtained inductively in this way is not asymptotically tight as the one in the above Theorem 3.3.

For the lower bound of $D(n;\tilde{a})$, where the k-tuple $\tilde{a}$ is given, we likewise define k integers $B_1, \ldots, B_k$ recursively by

$$\begin{aligned} B_1 &= 0, \\ B_i &= B_{i-1} + a_i\left(\frac{\gcd(\tilde{a}_{i-1})}{\gcd(\tilde{a}_i)} - 1\right) - 1, \text{ for } 2 \leq i \leq k. \end{aligned}$$

We now have the following lower bound.

Theorem 3.4 *For a k-tuple $\tilde{a}$ of positive integers let $d = \gcd(\tilde{a})$. If $n \geq B_k + k - 1$ and (1) has one solution, then*

$$D(n;\tilde{a}) \geq \frac{d}{\Pi\tilde{a}}\binom{n - B_k}{k-1}.$$

Proof. In the case when $k = 1$, the theorem is a tautology. In the case $k = 2$ the theorem is true by Observation 3.1.

We proceed by induction on k. As in the proof of Theorem 3.3 we have the recursive formula (4) where $x_k = x_k^* + id'/d$ is part of a nonnegative solution to (1). Let $c(i) = n - B_{k-1} - a_k(x_k^* + id'/d)$, and let M be the largest integer with $c(M) \geq k - 2$. Since $n \geq B_k + k - 1$, and the recursive definition of B_k in terms of B_{k-1}, we have that $c(0) \geq k - 2$ and hence $M \geq 0$. From (4) we therefore get the following inequality.

$$D(n;\tilde{a}) \geq \sum_{i=0}^{M} D(c(i) + B_{k-1}; \tilde{a}_{k-1}). \tag{6}$$

By induction hypothesis, (6) and Lemma 3.2 we get that

$$D(n;\tilde{a}) \geq \frac{d'}{\Pi\tilde{a}_{k-1}}\left(\sum_{i=0}^{M}\binom{c(i)}{k-2}\right) \geq \frac{d}{\Pi\tilde{a}}\binom{c(0)+1}{k-1} \geq \frac{d}{\Pi\tilde{a}}\binom{n - B_k}{k-1},$$

which completes the proof. □

Remark: The bonds given in Theorems 3.3 and 3.4 are asymptotically tight. These bounds are easily expressible and, more importantly, easy to compute from any given k-tuple $\tilde{a}$. Both A_k and B_k can be computed efficiently: Since $\gcd(\tilde{a}_i) = \gcd(a_i, \gcd(\tilde{a}_{i-1}))$ for each i, the problem of computing both A_k and B_k are essentially equally as expensive as the computation of the greatest common divisor of two numbers, both less than or equal to $m = \max_{1 \leq i \leq k} |a_i|$, exactly k times. In fact, using the *binary version* of the Euclidean Algorithm [14, p. 338], to compute $\gcd(u, v)$

where $u, v \leq m$, takes in the worst case $\lfloor \lg m \rfloor + 1$ arithmetic operations of additions, subtractions and divisions by 2 (no general divisions!) Hence, the computation of both A_k and B_k takes at most $O(k \lg m)$ arithmetic operations.

Combining the previous two results, Theorems 3.3 and 3.4, we have the following.

Corollary 3.5 *For a k-tuple $\tilde{a}$ of positive integers with* $\gcd(\tilde{a}) = 1$ *and for all $n \geq B_k + k - 1$ we have*

$$\frac{1}{\Pi\tilde{a}}\binom{n - B_k}{k-1} \leq D(n;\tilde{a}) \leq \frac{1}{\Pi\tilde{a}}\binom{n + A_k}{k-1}.$$

Remark: Note that in order to prove the above corollary for a general $\tilde{a}$ with $\gcd(\tilde{a}) = 1$, we had to prove slightly more general statements. A direct inductive proof of upper bound part of Corollary 3.5 can be obtained under the assumption that the first two factors of $\tilde{a}$ satisfy $\gcd(a_1, a_2) = 1$. However, $\gcd(\tilde{a}) = 1$ can, of course, hold without any of the $\binom{k}{2}$ pairs (a_i, a_j) being relatively prime.

We note that the ordering of the coefficients $a_1, \ldots, a_k$ in the k-tuple $\tilde{a}$ is relevant for computations of A_k and B_k. Different orderings can yield different values of A_k and B_k. In particular, if the two smallest coefficients are relatively prime, the coefficients should be listed in an increasing order. Specifically if $a_1 = 1$ (or before reordering, if $a_i = 1$ for some i) then we have that $A_i = |\tilde{a}_i| - 1$ and $B_i = 1 - i$ for all i. Hence, from Corollary 3.5 we have in that case

$$\frac{1}{\Pi\tilde{a}}\binom{n + k - 1}{k-1} \leq D(n;\tilde{a}) \leq \frac{1}{\Pi\tilde{a}}\binom{n + |\tilde{a}| - 1}{k-1},$$

for all $n \geq 0$. This coincides with (3) when $\tilde{a} = (1, 2, \ldots, k)$, so Corollary 3.5 is a generalization of (3).

4 Further Concrete Bounds

In the previous section we obtained concrete upper and lower bounds for $D(n;\tilde{a})$ which are valid for any n. Clearly it is not always the case that one needs the bounds for *all* n, but rather infinitely many n, which are evenly distributed among the nonnegative integers, when n tends to infinity. So in the case where we only need to use a concrete upper or lower bound for $D(n;\tilde{a})$ for infinitely many n, is it possible to get better bounds than those from the previous section?

In this final section we will obtain slightly better bounds for a general k-tuple $\tilde{a}$ which are better for infinitely many n, that are evenly distributed among the nonnegative integers. We will prove the following by a different approach than from previous section.

Theorem 4.1 *For any given k-tuple $\tilde{a}$ and for each nonnegative i, there are $n_1, n_2 \in \{i, \ldots, i + |\tilde{a}| - k\}$ such that*

$$D(n_1; \tilde{a}) \geq \frac{1}{\Pi\tilde{a}}\binom{n_1 + k - 1}{k - 1}, \tag{7}$$

$$D(n_2; \tilde{a}) \leq \frac{1}{\Pi\tilde{a}}\binom{n_2 + |\tilde{a}| - 1}{k - 1}. \tag{8}$$

Remark: Since we do not assume $a_i = 1$ for any i, Theorem 4.1 is tighter than the general bounds in Theorems 3.3 and 3.4.

Before proving Theorem 4.1, we will prove a lemma that has an easy proof, and Theorem 4.1 will then directly follow.

If k and n are positive integers, let $\mathcal{D}_k(n) = \{\tilde{y} : \sum y_i = n,\ y_i \geq 0\}$. Clearly we have $|\mathcal{D}_k(n)| = \binom{n+k-1}{k-1}$. Let $\tilde{a}$ be a fixed k-tuple, and let $\tilde{r}$ be a k-tuple with $r_i \in \{0, 1, \ldots, a_i - 1\}$ for each $i \in \{1, \ldots, k\}$. For a fixed $\tilde{r}$ we let $\mathcal{D}_{k;\tilde{r}}(n) = \{\tilde{y} : \sum y_i = n,\ y_i \geq 0,\ y_i \equiv r_i \pmod{a_i}\}$. We now have the disjoint union

$$\mathcal{D}_k(n) = \bigcup_{\tilde{r}} \mathcal{D}_{k;\tilde{r}}(n), \tag{9}$$

where the union runs over all possible $\tilde{r}$ for a given fixed $\tilde{a}$.

For each $\tilde{a}$ and nonnegative n let $\mathcal{D}(n; \tilde{a}) = \{\tilde{x} : \tilde{a} \cdot \tilde{x} = n,\ x_i \geq 0\}$, so $|\mathcal{D}(n; \tilde{a})|$ is here the denumerant $D(n; \tilde{a})$. Since $y_i = a_i x_i + r_i$ for each i provides a bijective correspondence between $\tilde{x} \in \mathcal{D}(n - |\tilde{r}|; \tilde{a})$ and $\tilde{y} \in \mathcal{D}_{k;\tilde{r}}(n)$, we have $|\mathcal{D}(n - |\tilde{r}|; \tilde{a})| = |\mathcal{D}_{k;\tilde{r}}(n)|$. By (9) we then have the following lemma.

Lemma 4.2 *For a nonnegative n and a fixed k-tuple $\tilde{a}$ we have*

$$\sum_{\tilde{r}} D(n - |\tilde{r}|; \tilde{a}) = \binom{n + k - 1}{k - 1},$$

where the sum runs over all the $\Pi\tilde{a}$ possibilities of $\tilde{r}$.

Proof. (Theorem 4.1:) Let n be given. By Lemma 4.2 we have that the average value of the $D(n - |\tilde{r}|; \tilde{a})$ among all the possible $\tilde{r}$, is precisely $\frac{1}{\Pi\tilde{a}}\binom{n+k-1}{k-1}$. Hence, for each fixed n, there are $\tilde{r}'$ and $\tilde{r}''$ such that

$$D(n - |\tilde{r}'|; \tilde{a}) \geq \frac{1}{\Pi\tilde{a}}\binom{n + k - 1}{k - 1},$$

$$D(n - |\tilde{r}''|; \tilde{a}) \leq \frac{1}{\Pi\tilde{a}}\binom{n + k - 1}{k - 1}.$$

Note that both $|\tilde{r}'|$ and $|\tilde{r}''|$ are $|\tilde{a}| - k$ or less, and hence both $n - |\tilde{r}'|$ and $n - |\tilde{r}''|$ are contained in the set $\{n - |\tilde{a}| + k, \ldots, n\}$. Hence, letting $i = n - |\tilde{a}| + k$, $n_1 = n - |\tilde{r}'|$ and $n_2 = n - |\tilde{r}''|$, Theorem 4.1 follows. $\square$

Corollary 4.3 *For a given nonnegative integer N let I_N be a set of N consecutive nonnegative integers. Further, let $\mathcal{P}_1(I_N)$ denote the set of all $n_1 \in I_N$ satisfying (7), and $\mathcal{P}_2(I_N)$ the set of all $n_2 \in I_N$ that satisfy (8). Then the density of both these sets satisfy*

$$|\mathcal{P}_1(I_N)|,\ |\mathcal{P}_2(I_N)| \geq \left\lfloor \frac{N}{|\tilde{a}| - k + 1} \right\rfloor .$$

REMARK: When $a_1 = 1$ in a k-tuple $\tilde{a}$ then both the upper and lower bounds in Theorem 4.1 coincide with the upper and lower bound given in Corollary 3.5.

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References

[1] John Riordan, *An Introduction to Combinatorial Analysis*, Wiley Publication in Statistics, John Wiley & Sons Inc., (1958).

[2] Petr Lisonek. *Denumerants and Their Approximations*, J. Comb. Math. Comb. Comput., **18**:225–232, (1995).

[3] Ronald L. Graham, Donald E. Knuth, Oren Patashnik. *Concrete Mathematics.* Addison–Wesley, second edition, (1998).

[4] George E. Andrews. *Partition Analysis, MacMahon's Algorithm.* Invited talk at the Thirty-Third Southeastern International Conference On Combinatorics, Graph Theory and Computing (CGTC-33) at Florida Atlantic University (FAU) in Boca Raton, Florida, March 7th (2002).

[5] George E. Andrews. *Personal communication.*

[6] G. H. Hardy, S. Ramanujan. *Asymptotic Formulae in Combinatory Analysis.* Proc. London. Math. Soc., **17**, No.2: 75–115, (1918).

[7] L. Comtet. *Advanced Combinatorics.* Dordrecht, Holland – Boston, U.S.A.: D. Reidel Publishing Company, (1974).

[8] Shinji Kuriki. *Une identite introduite par le denumerant.* TRU Mathematics, **13**, No.1: 53–54, (1977).

[9] Shinji Kuriki. *Sur une methode de calcul du denumerant.* TRU Mathematics, **14**, No.1: 47–48, (1978).

[10] Matthias Beck, Ira M. Gessel, Takao Komatsu. *The polynomial part of a restricted partition function related to the Frobenius problem.* Electron J. Combin., **8**, No.1: Note 7, 5 pp (2001).

[11] W. J. A. Colman. *An upper bound for the general restricted partition problem.* Fibonacci Quart., **25**, No.1: 38–44 (1987).

[12] J. H. van Lint, R. M. Wilson. *A Course in Combinatorics.* Cambridge University Press, (1992).

[13] E. T. Bell. *Interpolated Denumerants and Lambert Series.* American Journal of Mathematics, **65**, Issue 3: 382–386, (1943).

[14] Donald E. Knuth. *The Art of Computer Programming, Volume 2.* Addison–Wesley, third edition, (1998).