

SOLUTIONS TO PRACTICE PROBLEMS 12.1

1. Statements appear in (a), (b), (d), and (e). Both (c) and (f) are not statements.
2. (a) "Sally is the class president and Sally is an accounting major."
 (b) "Sally is not the class president."
 (c) "Sally is the class president or Sally is an accounting major."
 (d) "Sally is not the class president or Sally is an accounting major."
 (e) "Sally is the class president and Sally is not an accounting major."
 (f) "Sally is not the class president or Sally is not an accounting major."

12.2 Truth Tables

In this section we discuss how the truth values of the statements $p \wedge q$, $p \vee q$, and $\sim p$ depend on the truth values of p and q .

The simple statements will be denoted p , q , r , and so on. A *statement form* is an expression formed from simple statements and connectives according to the following rules.

A simple statement is a statement form.

If p is a statement form, $\sim p$ is a statement form.

If p and q are statement forms, then so are $p \wedge q$, $p \vee q$, and $p \rightarrow q$.

EXAMPLE 1 **Defining statement forms** Show that each of the following is a statement form according to the definition. Assume that p , q , and r are simple statements.

- (a) $(p \wedge \sim q) \rightarrow r$ (b) $\sim(p \rightarrow (q \vee \sim r))$

- Solution** (a) Since p , q , and r are simple statements, they are statement forms. Since q is a statement form, so is $\sim q$. Then $p \wedge \sim q$ is a statement form. Since r is a statement form, $(p \wedge \sim q) \rightarrow r$ is a statement form.
- (b) Since r is a statement form, so is $\sim r$. Since both q and $\sim r$ are statement forms, $q \vee \sim r$ is a statement form. However, p is also a statement form; thus $p \rightarrow (q \vee \sim r)$ is a statement form. The negation of a statement form is a statement form, so $\sim(p \rightarrow (q \vee \sim r))$ is also.

Now Try Exercise 1

The *propositional calculus* is the manipulation, verification, and simplification of logical statement forms. It allows us to see for example whether two different statement forms are logically equivalent in the sense that they must always have the same truth values.

The main mechanism for determining the truth values of statement forms is known as a *truth table*. It is also possible to use tree diagrams (see Chapter 6) to determine the truth values of statement forms. We begin with truth tables.

Consider any simple statement p . Then p has one of the two truth values T (TRUE) or F (FALSE). We can list the possible values of p in a table:

p
T
F

Clearly, $\sim p$ (the negation of p) is a form derived from p and has two possible values also. When p has the truth value T, $\sim p$ has the truth value F, and vice versa.

We represent the truth value of $\sim p$ in a truth table.

p	$\sim p$
T	F
F	T

The statement form $p \wedge q$ (p and q) is made up of two statements represented by p and q and the connective \wedge . The statement p can have one of the truth values T or F. Similarly, q can have one of the truth values T or F. Hence, by the multiplication principle, there are four possible pairs of truth values for p and q .

Their *conjunction*, $p \wedge q$, is true if and only if both p is true *and* q is true. The truth table for conjunction is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The *disjunction* of p and q , $p \vee q$ (p or q), is true if either p is true *or* q is true or both p and q are true. In English, the word “or” is ambiguous; we have to distinguish between the exclusive and inclusive “or.” For example, in the sentence “Ira will go to either Princeton or Stanford,” the word “or” is assumed to be exclusive since Ira will choose one of the schools, but not both. On the other hand, in the sentence, “Diana is smart or she is rich,” the “or” probably is inclusive since either Diana is smart, Diana is rich, or Diana is both smart and rich. The mathematical statement form $p \vee q$ uses the inclusive “or” and is unambiguous, as the truth table makes clear.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that T stands for TRUE and F stands for FALSE. Each line of the truth table corresponds to a combination of truth values for the components p and q . There are only two lines in the table for $\sim p$, while for the others there are four.

What we have described so far is a system of calculating the truth value of a statement form based on the truth of its component statement forms. The beauty of truth tables is that they can be used to determine the truth values of more elaborate statement forms by reapplying the basic rules.

EXAMPLE 2 Constructing a truth table Construct a truth table for the statement form

$$\sim(p \vee q).$$

Solution We write all the components of the statement form so that they can be evaluated for the four possible pairs of values for p and q . Note that we enter the truth values

for p and q in the same order as in the previous tables. It is a good idea to use this order all the time; that is,

p	q
T	T
T	F
F	T
F	F

The truth table for the form $\sim(p \vee q)$ contains a column for each calculation.

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

The third column represents the truth values of the disjunction of the first two columns, and the fourth column represents the negation of the third column. The statement form $\sim(p \vee q)$ is TRUE in one case: when both p and q are FALSE. Otherwise, $\sim(p \vee q)$ is FALSE.

Now Try Exercise 5

EXAMPLE 3 Tautology Construct a truth table for the statement form

$$\sim(p \wedge \sim q) \vee p.$$

Solution

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$\sim(p \wedge \sim q) \vee p$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	F	T	T
(1)	(2)	(3)	(4)	(5)	(6)

We fill in each column by using the rules already established. Column 3 is the negation of the truth values in column 2. Column 4 represents the conjunction of columns 1 and 3. Column 5 is the negation of the statement form whose values appear in column 4. Finally, column 6 is the disjunction of columns 5 and 1. We see that this statement form has truth value TRUE no matter what the truth values of the statements p and q are. Such a statement is called a tautology.

Now Try Exercise 7

A statement form that has truth value TRUE regardless of the truth values of the individual statement variables it contains is called a *tautology*.

A statement form that has truth value FALSE regardless of the truth values of the individual statement variables it contains is called a *contradiction*.

There is a more efficient way to prepare the truth table of Example 3. Use the same order for entering possible truth values of p and q . Put the statement form at the top of the table. Fill in columns under each operation as you need them, working from the inside out. We label the columns in the order in which we fill them in.

p	q	$\sim(p \wedge \sim q) \vee p$			
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T
(1)	(2)	(5)	(4)	(3)	(6)

We entered the values of p and q in columns 1 and 2 and then entered the values for $\sim q$ in column 3. We used the conjunction of columns 1 and 3 to fill in column 4, applied the negation to column 4 to get column 5, and the disjunction of columns 5 and 1 to get the values in column 6.

EXAMPLE 4 Constructing truth tables Construct the truth table for the statement form

$$(\sim p \vee \sim q) \wedge (p \wedge \sim q).$$

Solution Again, we will use the short form of the table, putting the statement form at the top of the table, and labeling each column in the order in which it was completed.

p	q	$(\sim p \vee \sim q) \wedge (p \wedge \sim q)$						
T	T	F	F	F	F	T	F	F
T	F	F	T	T	T	T	T	T
F	T	T	T	F	F	F	F	F
F	F	T	T	T	F	F	F	T
(1)	(2)	(3)	(5)	(4)	(7)	(6)		

There are some unlabeled columns. These were just recopied for convenience and clarity. It is not necessary to include them. Column 7, the conjunction of columns 5 and 6, tells us that the statement form is TRUE if and only if p is TRUE and q is FALSE.

Now Try Exercise 9

EXAMPLE 5 Compressing truth tables Find the truth table for the statement form

$$(p \vee q) \wedge \sim(p \wedge q).$$

Solution

p	q	$(p \vee q) \wedge \sim(p \wedge q)$			
T	T	T	F	F	T
T	F	T	T	T	F
F	T	T	T	T	F
F	F	F	F	T	F
(1)	(2)	(3)	(6)	(5)	(4)

Column 6, the conjunction of columns 3 and 5, gives the truth values of the statement form. ■

We note that the statement form in Example 5 can be read as “ p or q but not both p and q .” This is the *exclusive or* q . We will find it a useful connective and denote it by \oplus . The statement form $p \oplus q$ is true exactly when p is true or q is true but not both p and q are true. The truth table of $p \oplus q$ is

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

If there are three simple statements in a statement form and we denote them p , q , and r , then each could take on the truth values TRUE or FALSE. Hence, by the multiplication principle, there are $2 \times 2 \times 2 = 8$ different assignments to the three variables together. There are eight lines in a truth table for such a statement form. Again, in the truth table we will list all the T's and then all the F's for the first variable. Then we alternate TT followed by FF in the second column. Finally, for the third variable we alternate T with F. This ensures that we have listed all the possibilities and facilitates comparison of the final results.

EXAMPLE 6 **Constructing truth tables for three simple statements** Construct a truth table for $(p \vee q) \wedge [(p \vee r) \wedge \sim r]$.

Solution Note the order in which the T's and F's are listed in the first three columns. The columns are numbered in the order in which they were filled.

p	q	r	$(p \vee q) \wedge [(p \vee r) \wedge \sim r]$			
T	T	T	T	F	T	F F
T	T	F	T	T	T	T T
T	F	T	T	F	T	F F
T	F	F	T	T	T	T T
F	T	T	T	F	T	F F
F	T	F	T	F	F	F T
F	F	T	F	F	T	F F
F	F	F	F	F	F	F T
(1)	(2)	(3)	(4)	(8)	(5)	(7) (6)

The statement form has truth value TRUE if and only if p is TRUE and r is FALSE (regardless of the truth value of q).

Now Try Exercise 25 ■

The formal evaluation of the truth table of a statement form can help in deciding the truth value when particular sentences are substituted for the statement variables.

EXAMPLE 7 **Logically true or false** Let p denote the statement "London is the capital of England" and let q denote the statement "Venice is the capital of Italy." Determine the truth value of each of the following statements.

- (a) $p \wedge q$ (b) $p \vee q$ (c) $\sim p \wedge q$
 (d) $\sim p \vee \sim q$ (e) $\sim(p \wedge q)$ (f) $p \oplus q$

Solution First, we note that p has truth value TRUE and q has truth value FALSE. (Rome is the capital of Italy.) Thus the conjunction in (a) is FALSE. The disjunction in (b) is TRUE because p is TRUE. In (c), both $\sim p$ and q are false, so their conjunction is FALSE. Since $\sim q$ is TRUE, statement (d) is TRUE. Since (a) is FALSE, and the statement in (e) is its negation, (e) is TRUE. Since p is TRUE and q is FALSE, $p \oplus q$ is TRUE.

Now Try Exercise 37

Logic and Computer Languages In many computer languages, such as BASIC, the logical connectives are incorporated into programs that depend on logical decision making. The symbols are replaced by their original English words in the BASIC language; for example,

- AND \wedge
 OR \vee
 NOT \sim
 XOR \oplus (XOR stands for "exclusive or").

If the statements p and q are assigned truth values T and F, then the truth tables for the connectives AND, OR, XOR, and NOT are as shown in Fig. 1.

p	q	p AND q	p OR q	p XOR q	p	NOT p
T	T	T	T	F	T	F
T	F	F	T	T	F	T
F	T	F	T	T		
F	F	F	F	F		

Figure 1

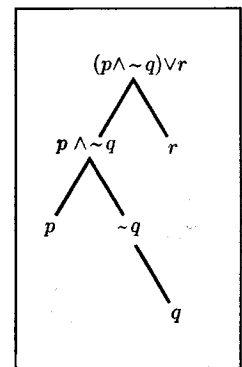


Figure 2

Use of a Tree to Represent a Statement Form Another method of describing a statement form is by using a mathematical structure called a *tree*. We have already seen tree diagrams in Chapter 6. The statement form $(p \wedge \sim q) \vee r$ is diagrammed in Fig. 2. We begin by writing the statement form at the top of the tree. Two branches are then required, one for each of the operands: $(p \wedge \sim q)$ and r . We continue until we get to forms involving only a simple statement alone. From $(p \wedge \sim q)$, we require two branches, one for each of the operands p and $\sim q$. Finally, one branch is needed to perform the negation of q .

If we have specific truth values for the statement variables, the tree can be used to determine the truth value of the statement form by entering the values for all the

variables and filling in truth values for the forms, working from the bottom to the top of the tree. Suppose that we are interested in the value of the statement form $(p \wedge \sim q) \vee r$ when p is TRUE, q is FALSE, and r is FALSE. Figure 3 shows how we fill in values at the nodes and proceed up the tree to its top for the truth value of the statement form.

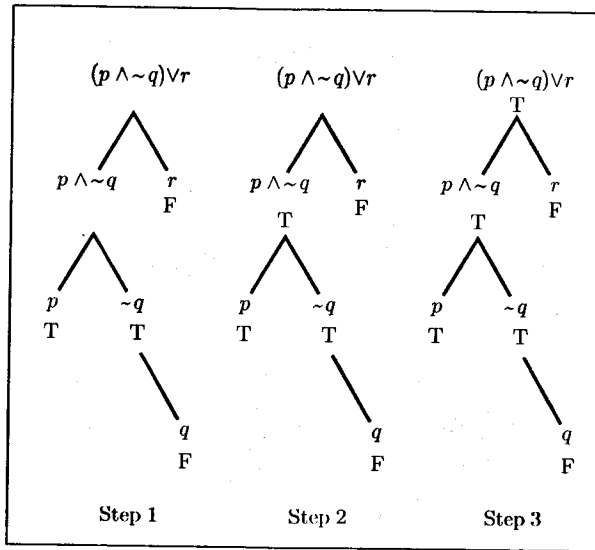


Figure 3

GC Graphing calculators can evaluate logical expressions involving the following four logical operators: AND, OR, XOR, NOT. See Fig. 4, which was obtained on the TI-83 by pressing $\boxed{2nd} \boxed{[TEST]}$. TRUE and FALSE are represented by the numbers 1 and 0, respectively. See Figs. 5 and 6. With the TI-89, the four operators along with the words *true* and *false*, can be either typed into the entry line, or by pressing $\boxed{2nd} \boxed{[MATH]}$ 8, or can be selected from the CATALOG menu.

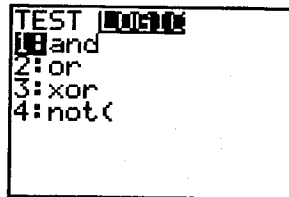


Figure 4

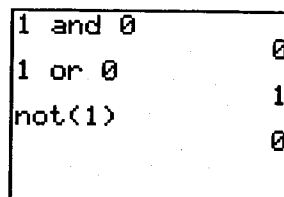


Figure 5

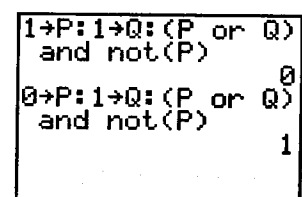


Figure 6

With a TI-83, an entire truth table can be displayed by creating lists of zeros and ones for the variables and attaching a formula containing logical operators to a list name (see Fig. 7). (The lists **P** and **Q** can be created from the home screen as in Fig. 8. A formula is attached to **L1** by entering a logical expression surrounded by quotation marks. The list names **LP** and **LQ** are displayed from the LIST menu.)

P	Q	$P \vee Q$	$\sim P$
1	1	1	0
1	0	1	0
0	1	1	1
0	0	0	1

L1 = "LP xor LQ"

Figure 7

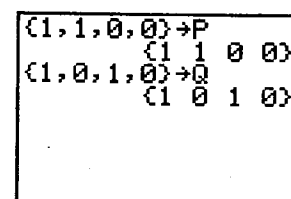


Figure 8

F1 Tools	F2 Plot Setup	F3 Cell Header	F4 Calc	F5 F6 Units	F7 Stat
DATA					
	c1	c2	c3		
1	true	true	false		
2	true	false	true		
3	false	true	true		
4	false	false	false		
c3=c1 xor c2					
MAIN		RAD AUTO		FUNC	

Figure 9

With the TI-89, an entire truth table can be created as shown in Fig. 9. Begin by creating a Data variable from the Data/Matrix editor found in the **APPS** menu. In the truth table of Fig. 9, the entries in the **c1** and **c2** columns were entered manually. The entries in the **c3** column appeared when **c3** was set to **c1 xor c2**.

ES An Excel spreadsheet can display an entire truth table for logical expressions involving AND, OR, and NOT. If x, y, \dots, z are addresses of cells containing the words TRUE or FALSE, then the truth values of the three logical functions are given in Table 1.

TABLE 1	
Function	Truth Value
AND(x, y, \dots, z)	(value of x) \wedge (value of y) $\wedge \dots \wedge$ (value of z)
OR(x, y, \dots, z)	(value of x) \vee (value of y) $\vee \dots \vee$ (value of z)
NOT(x)	\sim (value of x)

Figure 10 shows the truth table for the logical expression in Example 5. After the formula was entered into cell C1, the cell was selected and its fill handle was dragged down to C4.

C1		=AND(OR(A1,B1),NOT(AND(A1,B1)))				
	A	B	C	D	E	F
1	TRUE	TRUE	FALSE			
2	TRUE	FALSE	TRUE			
3	FALSE	TRUE	TRUE			
4	FALSE	FALSE	FALSE			

Figure 10. Truth table for Example 5.

PRACTICE PROBLEMS 12.2

- Construct the truth table for $(p \vee \sim r) \wedge q$.
- Construct the truth table for $p \oplus \sim q$.
- Let p denote "May follows April" and let q denote "June follows May." Determine the truth values of the following.

(a) $p \wedge \sim q$	(b) $\sim(p \vee \sim q)$
(c) $p \oplus q$	(d) $\sim[(p \wedge q) \oplus \sim q]$

EXERCISES 12.2

Show that the expressions in Exercises 1–4 are **statement forms** according to the definition. Assume $p, q,$ and r are simple statements.

- $\sim(p \wedge \sim r) \vee q$
- $(p \rightarrow \sim q) \wedge r$
- $(\sim p \vee r) \rightarrow (q \wedge r)$
- $\sim((p \wedge r) \rightarrow q)$

In Exercises 5–28, construct truth tables for the given statement forms.

- $p \wedge \sim q$
- $\sim(p \vee \sim q) \wedge (p \wedge \sim q)$
- $(p \vee \sim q) \vee \sim p$
- $(p \wedge q) \vee (p \wedge \sim q)$
- $\sim((p \vee q) \wedge (p \wedge q))$
- $(p \vee \sim q) \oplus (\sim p)$
- $p \oplus (\sim p \vee q)$
- $(p \oplus q) \wedge r$
- $(p \wedge \sim r) \vee q$
- $\sim(p \wedge r) \vee q$

- $\sim[(p \wedge r) \vee q]$
- $p \wedge \sim p$
- $p \vee \sim p$
- $p \oplus (q \vee r)$
- $(p \vee q) \wedge (p \vee r)$
- $(p \vee q) \wedge (p \vee \sim r)$
- $(p \vee q) \wedge \sim(p \vee q)$
- $(p \vee q) \wedge r$
- $\sim(p \vee q) \wedge r$
- $\sim(p \vee q) \wedge \sim r$
- $\sim p \vee (q \wedge r)$
- $p \wedge \sim p$
- $(p \vee q) \wedge \sim r$
- $p \vee (q \wedge r)$
- $(p \vee q) \wedge (p \vee \sim r)$
- $(\sim p \vee q) \wedge r$
- $\sim[(p \vee q) \wedge r]$
- $(p \oplus q) \oplus \sim(p \oplus q)$
- Compare the truth tables of $\sim p \vee \sim q$ and $\sim(p \wedge q)$.
- Compare the truth tables of $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$.
- Compare the truth tables of $\sim(p \oplus q)$ and $(p \wedge q) \vee \sim(p \vee q)$.

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32. How many possible truth tables can you construct for statement forms involving two variables, p and q ? (Hint: Consider the number of ways to complete the last column of a truth table for each of the four possible pairs of values for p and q .)
33. Compare the truth tables of $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$.
34. (The Symbol \oplus (NXOR)) Define the connective \oplus by the truth table

p	q	$p \oplus q$
T	T	T
T	F	F
F	T	F
F	F	T

Construct the truth tables of the following.

- (a) $p \oplus \sim q$ (b) $(p \oplus q) \oplus r$
 (c) $p \oplus (q \oplus r)$ (d) $\sim(p \oplus q) \wedge (p \oplus q)$

35. (The Sheffer Stroke (NAND)) Define $p|q$ by the truth table

p	q	$p q$
T	T	F
T	F	T
F	T	T
F	F	T

This connective is called the *Sheffer stroke* (or NAND). Construct truth tables for the following.

- (a) $p|p$ (b) $(p|p)|(q|q)$
 (c) $(p|q)|(p|q)$ (d) $p|((p|q)|q)$

36. Compare the truth tables of the connectives \sim , \vee , and \wedge with those constructed in Exercise 35. Do you see any similarities?
37. Let p denote "John Lennon was a member of the Beatles" and let q be the statement "The Beatles came from Spain." Determine the truth values of the following.
- (a) $p \vee \sim q$ (b) $\sim p \wedge q$
 (c) $p \oplus q$ (d) $\sim p \oplus q$
 (e) $\sim(p \oplus q)$ (f) $(p \vee q) \oplus \sim q$

38. Let p be the statement "There were 14 original American colonies" and let q denote "Utah was one of the original colonies." Determine the truth value of each of the following.

- (a) $p \vee q$ (b) $\sim p \vee q$
 (c) $\sim p \wedge \sim q$ (d) $p \oplus \sim q$

39. Let p be the statement "A rectangle has three sides" and q be the statement "A right angle has 90 degrees." Determine the truth value of each of the following.

- (a) $p \wedge \sim q$ (b) $\sim(p \oplus q)$
 (c) $p \wedge q$ (d) $\sim p \wedge \sim q$

40. (Using a Tree Diagram) Draw a tree for the statement form given in Exercise 5. Assuming p has truth value T and q has value F, use the tree to find the truth value of the statement form.

41. (Using a Tree Diagram) Draw a tree for the statement form given in Exercise 12. Assuming that p has truth value T, q has truth value F, and r has truth value F, use the tree to find the truth value of the statement form.

42. Repeat Exercise 40 for the statement in Exercise 9.

43. Repeat Exercise 41 for the statement in Exercise 13.

In Exercises 44 and 45, what value will be displayed when the Enter key is pressed?

44. $1 \rightarrow P: 0 \rightarrow Q: (P \text{ xor } Q)$
 $\text{and not}(Q)$

45. $1 \rightarrow P: 0 \rightarrow Q: (P \text{ and } Q)$
 $\text{or not}(P)$

Exercises 46–51 require the use of a graphing calculator.

In Exercises 46–49, use a graphing calculator to determine the value of the logical expression when (a) p is TRUE, q is TRUE, and (b) p is TRUE, q is FALSE.

46. $(p \oplus q) \wedge \sim q$ 47. $(p \vee q) \oplus \sim p$
 48. $\sim p \vee (p \wedge q)$ 49. $(p \wedge \sim q) \oplus p$

In Exercises 50 and 51, if your calculator can display a truth table, generate a truth table for the given expression.

50. $(p \vee \sim q) \oplus p$ 51. $\sim(p \oplus q) \vee (p \wedge q)$