CHAPTER 12

Logic

IN this chapter we introduce logic: the foundation for mathematics, coherent argument, and computing. The idea of logical argument was introduced in writing by Aristotle. For centuries his principles have formed the basis for systematic thought, communication, debate, law, mathematics, and science. More recently, the concept of a computing machine and the essence of programming are applications of the ancient ideas we present here. Also, new developments in mathematical logic have helped to promote significant advances in artificial intelligence.

12.1 Introduction to Logic

The building blocks of logic are statements, connectives, and the rules for calculating the truth or falsity of compound statements. We begin with statements.

A *statement* is a declarative sentence that is either true or false.
Here are some examples of statements:

1. George Washington was the first president of the United States.
2. The New York Knicks won the NBA basketball championship in 1989.
3. The number of atoms in the universe is 10^{75}.

4. Ronald Reagan knew what Oliver North was doing.

5. If x is a positive real number and $x^2 = 9$, then $x = 3$.

We know that statement 1 is true. Statement 2 is false. Statements 3 and 4 are either true or false, but we do not know which. Neither is both true *and* false. Statement 5 is true.

To better understand what a statement is, we list some nonstatements.

6. He is a real nice guy.

7. Do your homework!

8. If $x^2 = 9$, then $x = 3$.

9. Is this course fun?

Items 6 and 8 are not statements because the person "he" and the variable x are not specified. There are circumstances under which we would claim that item 6 is true and others under which we would claim that item 6 is false; clearly, it depends on who "he" is. Similarly, if x were a positive real number, item 8 would be true. If x were -3, item 8 would be false. Items 7 and 9 are not declarative sentences.

The *truth value* of a statement is either *true* or *false*. The problem of deciding the truth value of a declarative sentence will be dealt with in a simple manner, although it is far from simple. The assignment of a truth value to a statement may be obvious, as in the statement "George Washington was the first president of the United States." The truth value of the statement "The number of atoms in the universe is 10^{75}," on the other hand, is much more problematic.

Logic forms the basis for analyzing legal briefs, political rhetoric, and family discussions. It allows us to understand another's point of view as well as to expose weaknesses in an argument that lead to unfounded conclusions. We combine statements naturally when we speak, using connective words like "and" and "or." We state implications with the words "if" and "then." Frequently, we negate statements with the word "not." A *compound statement* is formed by combining statements using the words "and," "or," "not," or "if, then." A *simple statement* is a statement that is not a compound statement. A compound statement is analyzed as a combination of simple statements. Logic cannot be used to determine the truth value of a simple statement. However, if there is agreement on the truth value of simple statements that are components of a compound statement, the rules of mathematical logic determine the truth value of the compound statement. Its truth value is determined according to mathematical rules given in the next sections.

EXAMPLE 1 **Simple statements** Give the simple statements in each of the following compound statements.

(a) The number 6 is even and the number 5 is odd.

(b) Tom Jones does a term paper or takes the final exam.

(c) If England is in the Common Market, then the British eat Spanish oranges and Italian melons.

Solution (a) The simple statements are "The number 6 is even" and "The number 5 is odd."

(b) The simple statements are "Tom Jones does a term paper" and "Tom Jones takes the final exam."

(c) The simple statements are "England is in the Common Market," "The British eat Spanish oranges," and "The British eat Italian melons."

It is sometimes helpful to add English words to clarify the implied meaning in simple statements.

Now Try Exercise 11 ■

To be able to develop the rules of logic and logical argument, we need to deal with any logical statement, rather than specific examples. We use the letters p, q, r, and so on to represent simple statements. They are not really the statements themselves, but variables for which a statement may be substituted. For example, we can let p represent a statement in general. Then, if we wish, we can specify that for the moment p will represent the statement "The number 5 is odd." In this particular case, p has the truth value *true*. We might decide to let p represent the statement "There are 13 states in the United States today." Then the truth value of the statement p is *false*.

The use of these logical variables is similar to the use of variables x, y, z, and so on to represent unspecified numbers in algebra. In algebra, we manipulate the symbols (such as x, y, $+$, and the implied multiplication and exponentiation) in expressions according to specified rules to establish identities such as

$$(x + y)^2 = x^2 + 2xy + y^2,$$

which are true no matter which numbers are substituted for x and y. In logic, we will manipulate the symbols in compound statements (p, q, r, "and," "or," "not," and "if, then") to look for expressions that are true no matter which statements are substituted for p, q, r, and so on.

It is useful to be able to write a compound statement in terms of its component parts and to use symbols to represent the connectives. We use $\wedge$ to represent the word "and" and $\vee$ to represent the word "or."

EXAMPLE 2 **Symbolic "and"** Write the compound statement "Ina likes popcorn and Fred likes peanuts" in symbolic form.

Solution We first let p represent the statement "Ina likes popcorn" and we let q represent the statement "Fred likes peanuts." We use the symbol $\wedge$ to represent the word "and." Thus, we represent our compound statement symbolically as $p \wedge q$. ■

EXAMPLE 3 **Symbolic "or"** Write the compound statement "The United States trades with Japan or Germany trades with Japan" in symbolic form.

Solution We use the letter p to represent the statement "The United States trades with Japan" and the letter q to represent the statement "Germany trades with Japan." We use the symbol $\vee$ to represent the word "or." Then the compound sentence is of the form $p \vee q$. ■

The symbol $\sim$ represents "not" so that, with p as defined in the previous example, we would use $\sim p$ to represent the statement "The United States does not trade with Japan."

EXAMPLE 4 **Symbolic form of compound statements** Let p represent "Fred likes Cindy" and let q represent "Cindy likes Fred." Use the connectives $\wedge$, $\vee$, and $\sim$ to represent the following compound sentences:

(a) Fred and Cindy like each other.

(b) Fred likes Cindy but Cindy does not like Fred.

(c) Fred and Cindy dislike each other.

(d) Fred likes Cindy or Cindy likes Fred.

Solution (a) "Fred and Cindy like each other" should be rewritten as "Fred likes Cindy and Cindy likes Fred." This is expressed symbolically as $p \wedge q$. Statement (b) can be written $p \wedge \sim q$. Note that the word "but" here means "and." Statement (c) can be rewritten as "Fred dislikes Cindy and Cindy dislikes Fred." This is written symbolically as $\sim p \wedge \sim q$. The last statement (d) is simply $p \vee q$.

Now Try Exercise 17 ■

EXAMPLE 5 **Translating from symbols to English** Let p represent the statement "The interest rate is 10%" and let q be the statement "The Dow Jones average is over 10,000." Write the English statements corresponding to each of the following.

(a) $p \vee q$ (b) $p \wedge q$ (c) $p \wedge \sim q$ (d) $\sim p \vee \sim q$

Solution (a) The statement $p \vee q$ can be written, "The interest rate is 10% or the Dow Jones average is over 10,000."

(b) We write $p \wedge q$ as "The interest rate is 10% and the Dow Jones average is over 10,000."

(c) The statement $p \wedge \sim q$ becomes "The interest rate is 10% and the Dow Jones average is less than or equal to 10,000."

(d) The statement $\sim p \vee \sim q$ can be written as "The interest rate is not 10% or the Dow Jones average is less than or equal to 10,000."

Now Try Exercise 23 ■

The symbol $\rightarrow$ represents an implication. The statement $p \rightarrow q$ is read "p implies q" or "if p, then q." Thus, using the representations of Example 5, the English sentence "If the interest rate is 10%, then the Dow Jones average is over 10,000" can be written as $p \rightarrow q$.

EXAMPLE 6 **Translating from English to symbols** Let p denote the statement "The train stops in Washington" and let q denote the statement "The train stops in New York." Write the following statements in symbolic form.

(a) The train stops in New York and Washington.

(b) The train stops in Washington but not in New York.

(c) The train does not stop in New York.

(d) The train stops in New York or Washington.

(e) The train stops in New York or Washington but not in both.

(f) If the train stops in New York, then it does not stop in Washington.

Solution (a) $p \wedge q$. (b) $p \wedge \sim q$. (c) Negate q to get $\sim q$. (d) $p \vee q$. (e) $(p \vee q) \wedge \sim(p \wedge q)$. The parentheses make the statement clear. We discuss the need for them in Section 12.3. (f) $q \rightarrow \sim p$.

Now Try Exercise 25 ■

How are truth values assigned to compound statements? The assignment depends on the truth values of the simple statements and the connectives in the compound form. The rules are discussed in the next section.

PRACTICE PROBLEMS 12.1

1. Determine which of the following sentences are statements.
   (a) The earnings of IBM went up from 1993 to 1994.
   (b) The national debt of the United States is $6 trillion.
   (c) What an exam that was!
   (d) Abraham Lincoln was the sixteenth president of the United States.
   (e) Lexington is the capital of Kentucky or Albany is the capital of the United States.
   (f) When was the Civil War?
2. Let p denote the statement "Sally is the class president" and let q denote "Sally is an accounting major." Translate the symbolic statements into proper English.
   (a) $p \wedge q$ (b) $\sim p$ (c) $p \vee q$
   (d) $\sim p \vee q$ (e) $p \wedge \sim q$ (f) $\sim p \vee \sim q$

EXERCISES 12.1

In Exercises 1–15, determine which sentences are statements.

1. The number 3 is even.
2. The 1939 World's Fair was held in Miami.
3. The price of coffee depends on the rainfall in Brazil.
4. The Nile River flows through Asia.
5. What a way to go!
6. If snow falls on the Rockies, people are skiing in Aspen.
7. Why is the sky blue?
8. Moisture in the atmosphere determines the type of cloud formation.
9. No aircraft carrier is assigned to the Indian Ocean.
10. The number of stars in the universe is 10^{60}.
11. $x + 3 \geq 0$.
12. He is a brave fellow.
13. Let us pray.
14. The Louvre and the Metropolitan Museum of Art contain paintings by Leonardo da Vinci.
15. If a United States coin is fair, the chance of getting a head is $\frac{1}{3}$.

Find the simple statements in each of the compound statements in Exercises 16–19 and write the compound statement symbolically.

16. China is in Asia and Chicago is in North America.
17. The Phelps Library is either in New York or in Dallas.
18. Randy studies German on either Tuesday or Friday.
19. The Smithsonian Museum of Natural History has displays of rocks and bugs.

In Exercises 20 and 21, write the compound statements in symbolic form, using p and q for each of the simple statements.

20. The number 7 is odd and the number 14 is even.
21. No Amtrak trains go to Chicago or Cincinnati.
22. Let p denote the statement "Arizona has the largest U.S. Indian population" and let q denote the statement "Arizona is the site of the O.K. Corral." Write out the following statements in proper English sentences.
    (a) $\sim p$ (b) $\sim p \vee q$ (c) $\sim q \wedge p$
    (d) $p \vee q$ (e) $\sim p \wedge \sim q$ (f) $\sim(p \vee q)$
23. Let p denote the statement "Ozone is opaque to ultraviolet light" and let q denote the statement "Life on earth requires ozone." Write out the following statements in proper English.
    (a) $p \wedge q$ (b) $\sim p \vee q$ (c) $\sim p \vee \sim q$ (d) $\sim(\sim q)$
24. Let p denote the statement "Papyrus is the earliest form of paper" and let q denote "The papyrus reed is found in Africa." Put the following statements into symbolic form.
    (a) Papyrus is not the earliest form of paper.
    (b) The papyrus reed is not found in Africa or papyrus is not the earliest form of paper.
    (c) Papyrus is the earliest form of paper and the papyrus reed is not found in Africa.
25. Let p denote the statement "Florida borders Alabama" and let q denote the statement "Florida borders Mississippi." Put the following into symbolic form.
    (a) Florida borders Alabama or Mississippi.
    (b) Florida borders Alabama but not Mississippi.
    (c) Florida borders Mississippi but not Alabama.
    (d) Florida borders neither Alabama nor Mississippi.