SOLUTIONS TO PRACTICE PROBLEMS 12.2

1.

p	q	r	$(p \vee \sim r) \wedge q$		
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	T	F
(1)	(2)	(3)	(5)	(4)	(6)

2.

p	q	$p \oplus \sim q$	
T	T	T	F
T	F	F	T
F	T	F	F
F	F	T	T
(1)	(2)	(4)	(3)

3. Both p and q are TRUE. Thus (a) is FALSE. Item (b) is the negation of the statement $(p \vee \sim q)$, which is TRUE since p is TRUE. Hence, the statement in (b) is FALSE. Since both p and q are TRUE, $p \oplus q$ is FALSE. To analyze (d), we consider that $(p \wedge q)$ is TRUE and $\sim q$ is FALSE, so $[(p \wedge q) \oplus \sim q]$ is TRUE. Its negation is FALSE.

12.3 Implication

We are quite familiar with implications. For example, "If Jay is caught smoking in the restroom, he is suspended from school" is an implication. Note that, although Jay may avoid being caught smoking in the restroom, he still might be suspended for some other infraction. Symbolically, we can represent the statement "Jay is caught smoking in the restroom" by p, while we represent "Jay is suspended from school" by q. The implication is represented by the *conditional* connective $\rightarrow$ and we write $p \rightarrow q$. The truth table for the statement form using the conditional is as follows:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The *conditional* statement form $p \rightarrow q$ has truth value FALSE only when p has truth value TRUE and q has truth value FALSE.

In the preceding case, if p is TRUE, then "Jay is caught smoking in the restroom" is TRUE, while if q is FALSE, then "Jay is not suspended" is TRUE. The implication $p \rightarrow q$ is FALSE in this case.

We call p the *hypothesis* and q the *conclusion*. Note that if the hypothesis is FALSE, then the implication $p \rightarrow q$ is TRUE. There are several ways to read $p \rightarrow q$ in English:

1. p implies q.
2. If p, then q.
3. p only if q.
4. q, if p.
5. p is sufficient for q.
6. q is a necessary condition for p.

Note that $p \to q$ means that p is sufficient for q, so if you have a true p, then you get a true q. On the other hand, $q \to p$ means that p is necessary for q, so if you want a true q, then you must have a true p.

EXAMPLE 1 **Hypothesis and conclusion** In each of the following statements, determine the hypothesis and the conclusion.

(a) Bill goes to the party only if Greta goes to the party.

(b) Sue goes to the party if Craig goes to the party.

(c) For 6 to be even, it is sufficient that its square, 36, be even.

Solution (a) The statement is in the form of p only if q. Thus the hypothesis is "Bill goes to the party" and the conclusion is "Greta goes to the party." We could rewrite the statement as "If Bill goes to the party, then Greta goes to the party." The statement "If Bill goes to the party, then Greta goes to the party" seems to mean that Greta is following Bill around, whereas the statement "Bill goes to the party only if Greta goes to the party" makes the romance seem quite the opposite. However, this last statement should be interpreted as follows: If "Bill goes to the party" is true, then that means Greta must also have gone—for that was the ONLY reason he would go. The unemotional interpretation of the logical form shows the equivalence of the two statements.

(b) The statement is of the form q, if p. It has hypothesis "Craig goes to the party" and conclusion "Sue goes to the party."

(c) This is of the form p is sufficient for q, with the hypothesis p of the form "The square of the integer 6 is even" and the conclusion q of the form "The integer 6 is even."

Now Try Exercise 43 ■

EXAMPLE 2 **Truth value of an implication** Determine whether each of the following statements is true or false.

(a) If Paris is in France, then the Louvre is in Paris.

(b) If the Louvre is in Paris, then $2 + 3 = 7$.

(c) If $2 + 3 = 7$, then the Louvre is in Paris.

(d) If $2 + 3 = 7$, then Paris is in Spain.

(e) Paris is in Spain only if $2 + 3 = 7$.

Solution The statement (a) is TRUE because both the hypothesis and the conclusion are TRUE. The statement (b) is FALSE because the hypothesis is TRUE but the conclusion is FALSE. The last three statements are TRUE because the hypothesis in each is FALSE. Note that the last statement has hypothesis "Paris is in Spain" and conclusion "$2 + 3 = 7$."

Now Try Exercise 41 ■

EXAMPLE 3 **Using a truth table for an implication** Construct the truth table of the statement "If the president dies or becomes incapacitated, then the vice president becomes president."

Solution We let p be the statement "The president dies"; we let q be the statement "The president becomes incapacitated." We let r be the statement "The vice president becomes president." The statement form is $(p \vee q) \rightarrow r$. A truth table for the statement form is as follows:

p	q	r	$(p \vee q)$	$\rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	T
F	F	F	F	T
(1)	(2)	(3)	(4)	(5)

The statement is FALSE only in the cases where the president dies or becomes incapacitated but the vice president does not become president.

Now Try Exercise 11 ■

It is important to note that $p \rightarrow q$ is not the same as $q \rightarrow p$. We should not confuse the hypothesis and the conclusion in an implication. An example will demonstrate the difference between $p \rightarrow q$ and $q \rightarrow p$. Consider the implication "If the truck carries ice cream, then it is refrigerated." If the implication is in the form $p \rightarrow q$, then $q \rightarrow p$ is the implication "If the truck is refrigerated, then it carries ice cream." These implications need not have the same truth values. The first implication is probably true; the second need not be. The truth tables of $p \rightarrow q$ and $q \rightarrow p$ show the differences.

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

The implication $q \rightarrow p$ is called the *converse* of the statement $p \rightarrow q$. Thus, given the statement "If the Los Angeles Dodgers won the pennant, some games of the World Series are in California," its converse is "If some games of the World Series are in California, then the Los Angeles Dodgers won the pennant." While the original statement is true, the converse is not. There are situations in which the hypothesis of the converse is true, but the conclusion is false. Some games of the World Series may be in California because the San Francisco Giants, the Oakland Athletics, the San Diego Padres, or the Anaheim Angels, rather than the Los Angeles Dodgers, won the pennant.

EXAMPLE 4 **The conjunction of a statement and its converse** Construct the truth table for the conjunction of $p \rightarrow q$ and its converse. That is, find the truth table of the statement form $(p \rightarrow q) \wedge (q \rightarrow p)$.

Solution

p	q	$(p \to q)$	$\wedge$	$(q \to p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T
(1)	(2)	(3)	(5)	(4)

We note that the statement form has truth value TRUE whenever p and q have the same truth value: either both TRUE or both FALSE.

Now Try Exercise 45 ■

The statement form $(p \to q) \wedge (q \to p)$ is referred to as the *biconditional*, which we write as $p \leftrightarrow q$. The statement $p \leftrightarrow q$ is read as "p, if and only if q" or "p is necessary and sufficient for q." The truth table for the biconditional is as follows:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional is a statement form simply because it can be expressed as $(p \to q) \wedge (q \to p)$. In some sense, which we will discuss further in the next section, the biconditional expresses a kind of equivalence of p and q. That is, the biconditional statement form is TRUE whenever p and q are both TRUE or both FALSE. The biconditional is FALSE whenever p and q have different truth values.

Using connectives, we have seen how to string together several simple statements into compound and fairly complex statement forms. Such forms should not be confusing: What is the meaning of $\sim p \vee q$? Do we mean $\sim(p \vee q)$ or do we mean $(\sim p) \vee q$? To clarify matters and to avoid the use of too many parentheses, we define an order of *precedence* for the connectives. This dictates which of the connectives should be applied first.

> The order of precedence for logical connectives is
>
> $$\sim, \quad \wedge, \quad \vee, \quad \to, \quad \leftrightarrow.$$

One applies $\sim$ first, then $\wedge$, and so on. If there is any doubt, insert parentheses to clarify the statement form. Using the order of precedence, we see that $\sim p \vee q$ is $(\sim p) \vee q$.

EXAMPLE 5 **Order of precedence** Compare the truth tables of $(\sim p) \vee q$ and $\sim(p \vee q)$.

Solution

p	q	$(\sim p)$	$\vee\ q$	$\sim(p$	$\vee\ q)$
T	T	F	T	F	T
T	F	F	F	F	T
F	T	T	T	F	T
F	F	T	T	T	F
(1)	(2)	(3)	(4)	(6)	(5)

Note that columns 4 and 6 are different.

Now Try Exercise 17

EXAMPLE 6 **Order of precedence** Insert parentheses in the statement to show the proper order for the application of the connectives.

$$p \wedge q \vee r \rightarrow \sim s \wedge r$$

Solution Reading from the left, we apply the $\sim$ first, so $\sim s \wedge r$ becomes $(\sim s) \wedge r$. We then scan for the connective $\wedge$, since that is next in the precedence list. So $p \wedge q \vee r$ becomes $(p \wedge q) \vee r$. We apply the $\rightarrow$ last, and the statement can be written

$$[(p \wedge q) \vee r] \rightarrow [(\sim s) \wedge r]\,.$$

Now Try Exercise 19

Parentheses have highest priority in the precedence. Clearly,

$$p \wedge (q \vee (r \rightarrow \sim s)) \wedge r$$

has the same symbols as the statement of Example 6, but the parentheses have defined a different statement form.

That the statement form $p \vee q \vee r$ may be written either as $(p \vee q) \vee r$ or as $p \vee (q \vee r)$ is shown in another section. The statement form $p \wedge q \wedge r$ can be written as $(p \wedge q) \wedge r$ or as $p \wedge (q \wedge r)$, as is shown later.

Implications and Computer Languages The logical connective $\rightarrow$ is used in writing computer programs, although in some programming languages it is expressed with the words IF, THEN. Thus we might see an instruction of the form "IF $a = 3$ THEN LET $s = 0$." Strictly speaking, this is not a statement form, because "LET $s = 0$" is not a statement. However, that command means "the computer will set $s = 0$." Thus, if the value of a is 3, the computer sets s to 0. If $a \neq 3$, the program just continues. The instruction

"IF ... THEN ... ELSE"

allows for a branch in the program. For example,

"IF $a = 3$ THEN LET $s = 0$ ELSE LET $s = 1$"

assigns value 0 to s if $a = 3$ and assigns value 1 to s if $a \neq 3$.

EXAMPLE 7 **Implication and computing** For the given input values of A and B, use the program to determine the value of C. The asterisk ($*$) denotes multiplication.

IF $(\mathrm{A} * \mathrm{B}) + 6 \geq 10$
THEN LET $\mathrm{C} = \mathrm{A} * \mathrm{B}$
ELSE LET $\mathrm{C} = 10$

(a) A = −2, B = −7 (b) A = −2, B = 3 (c) A = 2, B = 2

Solution (a) We determine that $(A * B) + 6 = 14 + 6 = 20$. Hence, we set $C = 14$.
(b) In this case, $(A * B) + 6 = 0$. So we set $C = 10$.
(c) $(A * B) + 6 = 10$. Hence, $C = 4$.

Now Try Exercise 51 ■

Implication and Common Language The mathematical uses of the conditional and the biconditional are very precise. However, colloquial speech is rarely as precise. For example, consider the statement "Peter gets dessert only if he eats his broccoli." Technically, this means that if Peter gets dessert, he eats his broccoli. However, Peter and his parents probably interpret the statement to mean that Peter gets dessert if and only if he eats his broccoli. This imprecision in language is sometimes confusing; we make every effort to avoid confusion in mathematics by adhering to strict rules for using the conditional and biconditional.

PRACTICE PROBLEMS 12.3

1. Using

p: "A square is a rectangle"

and

q: "A rectangle has four sides,"

write out the statements in symbolic form. Name the hypothesis and the conclusion in each statement.

(a) A square is a rectangle if a rectangle has four sides.

(b) A rectangle has four sides if a square is a rectangle.

(c) A rectangle has four sides only if a square is a rectangle.

(d) A square is a rectangle is sufficient to show that a rectangle has four sides.

(e) For a rectangle to have four sides, it is necessary for a square to be a rectangle.

2. Let p denote the statement "There are 48 states in the United States," let q denote the statement "The American flag is red, white, and blue," and let r be the statement "Maine is on the East Coast." Determine the truth value of each of the following statement forms.

(a) $p \wedge q \rightarrow r$ **(b)** $p \vee q \rightarrow r$

(c) $\sim p \wedge q \leftrightarrow r$ **(d)** $p \wedge \sim q \rightarrow \sim r$

3. Use the program to assign a value to D in each of the cases listed.

```
LET C = A + B
IF (C > 0) OR (A > 0)
    THEN LET  D = 1
    ELSE LET  D = −1
```

(a) A = 4, B = 6 **(b)** A = 4, B = −6

(c) A = 4, B = −4 **(d)** A = −2, B = 4

(e) A = −6, B = 4 **(f)** A = −2, B = −2

EXERCISES 12.3

Construct a truth table for each of the statement forms in Exercises 1–15.

1. $p \rightarrow \sim q$
2. $p \vee (q \rightarrow \sim r)$
3. $(p \oplus q) \rightarrow q$
4. $(p \oplus q) \rightarrow r$
5. $(\sim p \wedge q) \rightarrow r$
6. $\sim(p \rightarrow q)$
7. $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
8. $p \oplus (q \rightarrow r)$
9. $(p \rightarrow q) \rightarrow r$
10. $p \rightarrow (q \rightarrow r)$
11. $\sim(p \vee q) \rightarrow (\sim p \wedge r)$
12. $p \rightarrow (p \oplus q)$
13. $(p \vee q) \leftrightarrow (p \wedge q)$
14. $[(\sim p \wedge q) \oplus q] \rightarrow p$
15. $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

For each of the Exercises 16–19, insert parentheses in accordance with the precedence of logical operations.

16. $p \wedge q \rightarrow p \vee \sim q$
17. $\sim p \wedge \sim q \rightarrow \sim p \wedge q$
18. $p \vee \sim q \wedge r \rightarrow p \vee r$
19. $\sim p \wedge \sim q \vee r \rightarrow \sim q \wedge r$

Let p be the TRUE statement "Abraham Lincoln was the sixteenth president of the United States" and let q be the FALSE statement "The battle of Gettysburg took place at the O.K. Corral." Determine the truth value of each of the statement forms in Exercises 20–29.

20. $p \rightarrow q$	21. $\sim p \rightarrow q$
22. $p \rightarrow \sim q$	23. $q \rightarrow p$
24. $p \leftrightarrow q$	25. $(p \oplus q) \rightarrow p$
26. $(p \vee q) \rightarrow q$	27. $(p \wedge \sim q) \rightarrow (\sim p \oplus q)$
28. $[p \wedge (\sim q \rightarrow \sim p)] \oplus q$	29. $p \rightarrow [p \wedge (p \oplus q)]$

In Exercises 30–37, write the statement forms in symbols using the conditional (→) or the biconditional (↔) connective. Name the hypothesis and the conclusion in each conditional form. Let p be the statement "Sally studied" and q be the statement "Sally passes."

30. If Sally studied, then Sally passes.
31. Sally studied if and only if Sally passes.
32. If Sally passes, then Sally studied
33. Sally passes only if Sally studied.
34. That Sally studied is sufficient for Sally to pass.
35. Sally's studying is necessary for Sally to pass.
36. Sally passes implies that Sally studied.
37. If Sally did not study, then Sally does not pass.

In Exercises 38–42, let p denote the TRUE statement "The die is fair" and let q be the TRUE statement "The probability of a 2 is $\frac{1}{6}$." Write each of the statement forms in symbols, name the hypothesis and the conclusion in each, and determine whether the statement is TRUE or FALSE.

38. If the die is fair, the probability of a 2 is $\frac{1}{6}$.
39. If the die is not fair, the probability of a 2 is not $\frac{1}{6}$.
40. The probability of a 2 is $\frac{1}{6}$ only if the die is fair.
41. The die is not fair if the probability of a 2 is not $\frac{1}{6}$.
42. The die is fair implies that the probability of a 2 is $\frac{1}{6}$.
43. Give the hypothesis and the conclusion in each statement.
    (a) Healthy people live a long life.
    (b) The train stops at the station only if a passenger requests it.
    (c) For the azalea plant to grow, it is necessary that it be exposed to sunlight.
    (d) Only if Jane goes to the store, I will go to the store.
44. State the hypothesis and conclusion in each statement form.
    (a) Copa Beach is crowded if the weather is hot and sunny.
    (b) Our team wins a game only if I carry a rabbit's foot.
    (c) If I carry a rabbit's foot, our team wins a game.
    (d) Ivy is green is sufficient for it to be healthy.
45. State the converse of each of the following statements.
    (a) City Sanitation collects the garbage if the mayor calls.
    (b) The price of beans goes down only if there is no drought.
    (c) If goldfish swim in Lake Erie, Lake Erie is fresh water.
    (d) If tap water is not salted, then it boils slowly.
46. State the converse of each of the following statements.
    (a) If Jane runs 20 miles, Jane is tired.
    (b) Cindy loves Fred only if Fred loves Cindy.
    (c) Jon cashes a check if the bank is open.
    (d) Errors are clear only if the documentation is complete.
    (e) Sally's eating the vegetables is a necessary condition for Sally's getting dessert.
    (f) Sally's eating the vegetables is sufficient for Sally's getting dessert.
47. Determine the output value of A in the program for the given input values of X and Y.

```
LET Z = X + Y
IF (Z ≠ 0) AND (X > 0)
   THEN LET A = 6
   ELSE LET A = 4
```

(a) X = 0, Y = 0	(b) X = 8, Y = −8
(c) X = −3, Y = 3	(d) X = −3, Y = 8
(e) X = 8, Y = −3	(f) X = 3, Y = −8

48. For the given input values of A and B, use the program to find the output values of X.

```
LET C = A + B
IF ((A > 0) OR (B > 0)) AND (C > 0)
   THEN LET X = 100
   ELSE LET X = −100
```

(a) A = 2, B = 2	(b) A = 2, B = −2
(c) A = 2, B = −5	(d) A = −2, B = −5
(e) A = −5, B = 3	(f) A = −5, B = 8

49. For the input values of A and B, use the program to determine the value of Y.

```
LET C = A * B
IF ((C ≥ 10) AND (A < 0)) OR (B < 0)
   THEN LET Y = 7
   ELSE LET Y = 0
```

(a) A = −2, B = −6 (b) A = −1, B = −6
(c) A = −2, B = 6 (d) A = 6, B = −1
(e) A = 4, B = 3 (f) A = 3, B = 1

50. For the given input values of X and Y, use the program to determine the output value of Z.

```
IF Y ≠ 0
    THEN LET Z = X/Y
    ELSE LET Z = −1,000,000
```

(a) X = 6, Y = 2 (b) X = 10, Y = 5
(c) X = 5, Y = 10 (d) X = 0, Y = 10
(e) X = 10, Y = 0 (f) X = −1,000,000, Y = 1

51. For the given input values of A and B, use the program to determine the value of C.

```
IF ((A < 0) AND (B < 0)) OR (B ≥ 6)
    THEN LET C = (A * B) + 4
    ELSE LET C = 0
```

(a) A = −1, B = −2 (b) A = −2, B = 8
(c) A = −2, B = 3 (d) A = 3, B = −2
(e) A = 3, B = 8 (f) A = 3, B = −3

52. For the given input values for A and B, use the program to determine the output value of C.

```
IF ((A ≥ 5) OR (B ≥ 5)) AND (B ≤ 10)
    THEN LET C = A − B
    ELSE LET C = −30
```

(a) A = 7, B = 7 (b) A = 7, B = 3
(c) A = 7, B = 12 (d) A = 4, B = 7
(e) A = 4, B = 3 (f) A = 4, B = 12

53. For the given input values of A and B, find the value of X in the program.

```
LET C = A − B
IF (C < 0) OR (B < 0)
    THEN LET D = 5 * C
    ELSE LET D = 0
LET X = D + 3
```

(a) A = 0, B = 0 (b) A = 6, B = 3
(c) A = −5, B = 3 (d) A = 3, B = 5
(e) A = 5, B = −3 (f) A = −5, B = −3

54. We let S be the weekly salary for a consultant who works H hours at a rate of R dollars per hour. We take a deduction of 4% on salaries up to $1000 per week and deduct the maximum of $40 from salaries above $1000. Use the following program to find the pay P for each of the consultants given the rate and the number of hours worked in a week.

```
LET S = R * H
IF S < 1000
    THEN LET D = .04 * S
    ELSE LET D = 40
LET P = S − D
```

(a) R = 10, H = 40 (b) R = 20, H = 40
(c) R = 25, H = 20 (d) R = 25, H = 40
(e) R = 30, H = 40 (f) R = 35, H = 40

SOLUTIONS TO PRACTICE PROBLEMS 12.3

1. (a) This is of the form $q \rightarrow p$ with hypothesis q and conclusion p. Statement (b) is of the form $p \rightarrow q$, where p is the hypothesis and q is the conclusion. Statement (c) is of the form $q \rightarrow p$, since it states that "If a rectangle has four sides, then a square is a rectangle." Note that "only if" signals the clause that is the conclusion. In (c), q is the hypothesis and p is the conclusion. Statement (d) is of the form $p \rightarrow q$ with hypothesis p and conclusion q. Statement (e) is the converse of (d) and is of the form $q \rightarrow p$ with hypothesis q and conclusion p.