The Discrete Haar transform (DHT).

(a) Let $\{c_0(k)\}_{k=0}^{2^N-1}$ be given. The key to defining the DHT is to interpret this data appropriately.

(b) We assume that there is an underlying function $f(x)$ on $[0,1]$ such that $c_0(k) = \langle f, p_{N,k} \rangle$. Since this data is all we know about $f(x)$, we assume that

$$f(x) = \sum_{k=0}^{2^N-1} c_0(k)\, p_{N,k}(x),$$

that is, that f is a scale N dyadic step function.

(c) Let us define

$$c_j(k) = \langle f, p_{N-j,k} \rangle \text{ and } d_j(k) = \langle f, h_{N-j,k} \rangle$$

for $1 \leq j \leq N$. The full Haar expansion of f in this case will be

$$f(x) = c_N(0)\, p_{0,0}(x) + \sum_{j=1}^{N} \sum_{k=0}^{2^{N-j}-1} d_j(k)\, h_{N-j,k}(x).$$

Note the reversal of indices!

(d) Lemma. For each ℓ, k,

$$p_{\ell,k}(x) = (1/\sqrt{2})\,(p_{\ell+1,2k}(x) + p_{\ell+1,2k+1}(x)),$$

$$h_{\ell,k}(x) = (1/\sqrt{2})\,(p_{\ell+1,2k}(x) - p_{\ell+1,2k+1}(x)).$$

Then

$$\begin{aligned} & c_j(k) \\ = \;& \langle f, p_{N-j,k}\rangle \\ = \;& \frac{1}{\sqrt{2}}\,\langle f, p_{N-j+1,2k}\rangle + \frac{1}{\sqrt{2}}\,\langle f, p_{N-j+1,2k+1}\rangle \\ = \;& \frac{1}{\sqrt{2}}\,c_{j-1}(2k) + \frac{1}{\sqrt{2}}\,c_{j-1}(2k+1), \end{aligned}$$

and

$$d_j(k) = \frac{1}{\sqrt{2}}\,c_{j-1}(2k) - \frac{1}{\sqrt{2}}\,c_{j-1}(2k+1).$$

(e) In matrix notation,

$$\begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix},$$

and

$$\begin{pmatrix} c_{j-1}(2k) \\ c_{j-1}(2k+1) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_j(k) \\ d_j(k) \end{pmatrix}.$$

So the DHT is reversible!

(f) $L = 2^{N-j+1}$ and define the $(L/2) \times L$ matrices H_L and G_L by

$$H_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & & \cdots & & 0 & 1 & 1 \end{pmatrix},$$

$$G_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & & \cdots & & 0 & 1 & -1 \end{pmatrix}.$$

Let $\mathbf{c}_j = [c_j(0)\ c_j(1)\ \cdots\ c_j((L/2)-1)]^T$ and $\mathbf{d}_j = [d_j(0)\ d_j(1)\ \cdots\ d_j((L/2)-1)]^T$. Then

$$\begin{pmatrix} \mathbf{c}_j \\ \mathbf{d}_j \end{pmatrix} = \begin{pmatrix} H_L\, \mathbf{c}_{j-1} \\ G_L\, \mathbf{c}_{j-1} \end{pmatrix} = \begin{pmatrix} H_L \\ G_L \end{pmatrix} \mathbf{c}_{j-1}.$$

(h) Definition. An $n \times n$ matrix U is an *orthogonal* or *unitary matrix* if its columns (or rows) form an orthonormal system in n-dimensional space. If U is real-valued, then $U^T U = I$ (where U^T is the transpose of U) and if U is complex valued, then $U^* U = I$ (where U^* is the conjugate transpose of U).

It is not hard to see that $\begin{pmatrix} H_L \\ G_L \end{pmatrix}$ is an orthogonal real-valued matrix. Therefore,

$$\begin{aligned} \mathbf{c}_{j-1} &= \begin{pmatrix} H_L \\ G_L \end{pmatrix}^T \begin{pmatrix} \mathbf{c}_j \\ \mathbf{d}_j \end{pmatrix} \\ &= \begin{pmatrix} H_L^T & G_L^T \end{pmatrix} \begin{pmatrix} \mathbf{c}_j \\ \mathbf{d}_j \end{pmatrix} \\ &= H_L^T \mathbf{c}_j + G_L^T \mathbf{d}_j. \end{aligned}$$