Properties of the Haar series.

1. Localization.

(a) Suppose that a function $f(x)$ vanishes outside a small subinterval (a, b) of $[0, 1)$. Then the Haar coefficient $\langle f, h_{j,k}\rangle$ will be zero if $I_{j,k}$ does not intersect (a, b). Hence if (a, b) is small, most of the Haar coefficients of $f(x)$ will vanish.

(b) Specifically, fix $j \geq 0$. Then $\langle f, h_{j,k}\rangle = 0$ if

$$2^{-j}(k+1) \leq a \qquad \text{or} \qquad 2^{-j}k \geq b$$

which is the same as

$$k \leq 2^j a - 1 \qquad \text{or} \qquad k \geq 2^j b.$$

Therefore, if $\langle f, h_{j,k}\rangle \neq 0$, then

$$2^j a - 1 < k < 2^j b.$$

Let N_j be the number of integers k satisfying the above. Then

$$2^j(b-a) \leq N_j < 2^j(b-a) + 2.$$

(c) There are 2^j Haar coefficients at scale j. The proportion of possibly nonzero Haar coefficients satisfies

$$(b-a) \leq \frac{N_j}{2^j} < (b-a) + 2^{-j+1}$$

so that

$$\lim_{j \to \infty} \frac{N_j}{2^j} = (b-a).$$

We conclude that the fraction of possibly nonzero Haar coefficients for a function vanishing outside an interval is approximately proportional to the length of that interval.

Behavior near jumps.

(a) $f(x) = \mathbf{1}_{[0,11/16)}(x)$. Note first that $\langle f, p_{0,0}\rangle = 11/16$ and $\langle f, h_{j,k}\rangle = 0$ whenever $I_{j,k} \subseteq [0, 11/16)$ or $I_{j,k} \subseteq [11/16, 1)$. This is true for every $j \geq 4$ and all k.

$$\langle f, h_{0,0}\rangle = 5 \cdot 2^{-4} = 5/16$$
$$\langle f, h_{1,1}\rangle = 3 \cdot 2^{-7/2} = (6/16)2^{-1/2}$$
$$\langle f, h_{2,2}\rangle = 2^{-3} = (4/16)2^{-1}$$
$$\langle f, h_{3,5}\rangle = 2^{-5/2} = (8/16)2^{-3/2}.$$

(b) Let $f(x) = \mathbf{1}_{[0,2/3)}(x)$. $\langle f, h_{j,k}\rangle \neq 0$ if and only if $2/3 \in I_{j,k}$. At each scale j there is exactly one nonzero Haar coefficient.

(j,k)	$\lvert\langle f, h_{j,k}\rangle\rvert$	$2^{j/2}\,\lvert\langle f, h_{j,k}\rangle\rvert$
$(0,0)$	.33333	.33333
$(1,1)$	.23570	.33333
$(2,2)$	.16667	.33333
$(3,5)$	.11785	.33333
$(4,10)$	.083333	.33333
$(5,21)$	.058926	.33333
$(6,42)$	.041667	.33333
$(7,85)$	.029463	.33333
$(8,170)$	.020833	.33333

(c) Let's examine more closely this behavior near jumps. Suppose that $f(x)$ is defined on $[0,1]$ with a jump at $x_0 \in (0,1)$, and that f' and f'' are continuous on $[0,x_0]$ and $[x_0,1]$. Fix, j and k and let $x_{j,k} = 2^{-j}(k+1/2)$ be the midpoint of $I_{j,k}$.

Case1: $x_0 \notin I_{j,k}$. By Taylor's formula,

$$\begin{aligned} f(x) &= f(x_{j,k}) + f'(x_{j,k})(x - x_{j,k}) \\ &\quad + \frac{1}{2} f''(\xi_{j,k})(x - x_{j,k})^2, \end{aligned}$$

for some $\xi_{j,k} \in I_{j,k}$.

$$\begin{aligned} \langle f, h_{j,k} \rangle &= \int_{I_{j,k}} f(x)\, h_{j,k}(x)\, dx \\ &= f(x_{j,k}) \int_{I_{j,k}} h_{j,k}(x)\, dx \\ &\quad + f'(x_{j,k}) \int_{I_{j,k}} h_{j,k}(x)(x - x_{j,k})\, dx \\ &\quad + \frac{1}{2} \int_{I_{j,k}} h_{j,k}(x)\, (x - x_{j,k})^2\, f''(\xi_{j,k})\, dx \\ &= -\frac{1}{4} f'(x_{j,k}) 2^{-3j/2} + r_{j,k}. \end{aligned}$$

Estimating the remainder,

$$\begin{aligned}|r_{j,k}| &= \frac{1}{2}\left|\int_{I_{j,k}} h_{j,k}(x)\,(x-x_{j,k})^2\, f''(\xi_{j,k})\,dx\right| \\ &\le \frac{1}{2}\,2^{j/2}\max_{x\in I_{j,k}}|f''(x)|\int_{I_{j,k}}(x-x_{j,k})^2\,dx \\ &= \frac{1}{24}\max_{x\in I_{j,k}}|f''(x)|2^{-5j/2}.\end{aligned}$$

Hence for large j,

$$|\langle f, h_{j,k}\rangle| \approx \frac{1}{4}|f'(x_{j,k})|2^{-3j/2}.$$

Case 2: $x_0 \in I_{j,k}$. Assume that $x_0 \in I^{\ell}_{j,k}$. By Taylor's formula

$$\begin{aligned}f(x) &= f(x_0-) + f'(\xi_-)(x-x_0), \quad x\in[0,x_0) \\ f(x) &= f(x_0+) + f'(\xi_+)(x-x_0), \quad x\in(x_0,1]\end{aligned}$$

some $\xi_- \in [x, x_0]$ and $\xi_+ \in [x_0, x]$.

$$
\begin{aligned}
&\langle f, h_{j,k}\rangle \\
&= \int_{I_{j,k}} f(x)\, h_{j,k}(x)\, dx \\
&= 2^{j/2} \int_{2^{-j}k}^{x_0} f(x_0-)\, dx \\
&\quad +2^{j/2} \int_{x_0}^{2^{-j}(k+1/2)} f(x_0+)\, dx \\
&\quad -2^{j/2} \int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} f(x_0+)\, dx + \epsilon_{j,k} \\
&= 2^{j/2}\, (x_0 - 2^{-j}k)\, (f(x_0-) - f(x_0+)) + \epsilon_{j,k},
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_{j,k} &= \int_{2^{-j}k}^{x_0} f'(\xi_-)(x-x_0)\, h_{j,k}(x)\, dx \\
&\quad + \int_{x_0}^{2^{-j}(k+1)} f'(\xi_+)(x-x_0)\, h_{j,k}(x)\, dx.
\end{aligned}
$$

Estimating the remainder,

$$
\begin{aligned}
|\epsilon_{j,k}| &\leq \max_{t\in I_{j,k}\setminus\{x_0\}} |f'(t)| \int_{I_{j,k}} |x-x_0|\, |h_{j,k}(x)|\, dx \\
&\leq \max_{t\in I_{j,k}\setminus\{x_0\}} |f'(t)|\, 2^{j/2} \int_{I_{j,k}} |x-x_0|\, dx \\
&\leq \frac{1}{4} \max_{t\in I_{j,k}\setminus\{x_0\}} |f'(t)|\, 2^{-3j/2}.
\end{aligned}
$$

Hence for large j,

$$|\langle f, h_{j,k}\rangle| \approx 2^{j/2}\,|x_0 - 2^{-j}k|\,|f(x_0-) - f(x_0+)|.$$

Generically, x_0 will be fairly close to $x_{j,k}$ so that $|x_0 - 2^{-j}k| \approx (1/4)\,2^{-j}$. Thus, for large j,

$$|\langle f, h_{j,k}\rangle| \approx \frac{1}{4}|f(x_0-) - f(x_0+)|\,2^{-j/2}.$$

(d) The Haar coefficients of a continuous function will be $O(2^{-3j/2})$ for large j. See Example 5.25(d) and Section 5.4.3 in the book.