The Dyadic Intervals.

Definition. For each pair of integers j, $k \in \mathbf{Z}$, define the interval $I_{j,k}$ by

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)).$$

The collection of all such intervals is called the collection of *dyadic subintervals of* $\mathbf{R}$.

Lemma. Given j_0, k_0, j_1, $k_1 \in \mathbf{Z}$, with either $j_0 \neq j_1$ or $k_0 \neq k_1$, then either

(a) $I_{j_1,k_1} \cap I_{j_0,k_0} = \emptyset$,

(b) $I_{j_1,k_1} \subseteq I_{j_0,k_0}$, or

(c) $I_{j_0,k_0} \subseteq I_{j_1,k_1}$.

In the latter two cases, the smaller interval is contained in either the right half or left half of the larger.

Definition. We write $I_{j,k} = I^{\ell}_{j,k} \cup I^{r}_{j,k}$, where $I^{\ell}_{j,k}$ and $I^{r}_{j,k}$ are dyadic intervals at scale $j+1$, to denote the left half and right half of the interval $I_{j,k}$. In fact, $I^{\ell}_{j,k} = I_{j+1,2k}$ and $I^{r}_{j,k} = I_{j+1,2k+1}$.

Dyadic Step Functions.

Definition. A *dyadic step function* is a step function $f(x)$ with the property that for some $j \in \mathbf{Z}$, $f(x)$ is constant on all dyadic intervals $I_{j,k}$, $k \in \mathbf{Z}$. We say in this case that $f(x)$ is a scale j dyadic step function.

Remark. (a) For each $j \in \mathbf{Z}$, the collection of all scale j dyadic step functions is a linear space. This means that if $f(x)$ and $g(x)$ are scale j dyadic step functions, then so is $a\, f(x) + b\, g(x)$ for any constants a and b.

(b) If $f(x)$ is a scale j dyadic step function on an interval I, then $f(x)$ is also a scale j' dyadic step function on I for any $j' \geq j$.

Dilation and Translation.

Definition Given $a > 0$, the *dilation operator*, D_a is given by

$$D_a f(x) = a^{1/2} f(ax).$$

Given $b \in \mathbf{R}$, the *translation operator*, T_b is given by

$$T_b f(x) = f(x - b).$$

Remark. Note that if $a > 1$, then $D_a f(x)$ is a "narrowed down" version of $f(x)$, and if $0 < a < 1$, then $D_a f(x)$ is a "spread out" version of $f(x)$.

Lemma. For every $f(x)$ and $g(x)$ defined on $\mathbf{R}$, and for every $a > 0$, $b \in \mathbf{R}$, the following hold.

(a) $D_a T_b f(x) = a^{1/2} f(ax - b)$.

(b) $D_a T_b f(x) = T_{a^{-1}b} D_a f(x)$.

The Haar system.

Definition. Let $p(x) = \mathbf{1}_{[0,1)}(x)$. (Note: If E is any subset of $\mathbf{R}$ then $\mathbf{1}_E(x)$ is the function that is 1 if $x \in E$ and 0 if $x \notin E$. It is referred to as the *characteristic function* or *indicator function* of E).

For each j, $k \in \mathbf{Z}$, define

$$p_{j,k}(x) = 2^{j/2}\, p(2^j x - k) = D_{2^j} T_k p(x).$$

The collection $\{p_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is referred to as the system of *Haar scaling functions*. For each $j \in \mathbf{Z}$, the collection $\{p_{j,k}(x)\}_{k\in\mathbf{Z}}$ is referred to as the system of *scale j Haar scaling functions*.

Let $h(x) = \mathbf{1}_{[0,1/2)}(x) - \mathbf{1}_{[1/2,1)}(x)$, and for each j, $k \in \mathbf{Z}$, define

$$h_{j,k}(x) = 2^{j/2}\, h(2^j x - k) = D_{2^j} T_k h(x).$$

The collection $\{h_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is referred to as the *Haar system on* $\mathbf{R}$. For each $j \in \mathbf{Z}$, the collection $\{h_{j,k}(x)\}_{k\in\mathbf{Z}}$ is referred to as the system of *scale j Haar functions*.

Remark. (a) For each j, $k \in \mathbf{Z}$,

$$p_{j,k}(x) = 2^{j/2}\,\mathbf{1}_{I_{j,k}}(x),$$

$$\begin{aligned} h_{j,k}(x) &= 2^{j/2}\,(\mathbf{1}_{I^{\ell}_{j,k}}(x) - \mathbf{1}_{I^{r}_{j,k}}(x)) \\ &= 2^{j/2}\,(\mathbf{1}_{I_{j+1,2k}}(x) - \mathbf{1}_{I_{j+1,2k+1}}(x)). \end{aligned}$$

Both $p_{j,k}(x)$ and $h_{j,k}(x)$ are supported on the interval $I_{j,k}$ and neither one vanishes on that interval. We associate to each interval $I_{j,k}$ the pair of functions $p_{j,k}(x)$ and $h_{j,k}(x)$.

(b) For each j, $k \in \mathbf{Z}$, $p_{j,k}(x)$ is a scale j dyadic step function (hence also a scale $j+1$ dyadic step function), and $h_{j,k}(x)$ is a scale $j+1$ dyadic step function.

(c) For each j, $k \in \mathbf{Z}$,

$$\int_{\mathbf{R}} p_{j,k}(x)\,dx = \int_{I_{j,k}} p_{j,k}(x)\,dx = 2^{-j/2},$$

$$\int_{\mathbf{R}} |p_{j,k}(x)|^2\,dx = \int_{I_{j,k}} |p_{j,k}(x)|^2\,dx = 1,$$

$$\int_{\mathbf{R}} h_{j,k}(x)\,dx = \int_{I_{j,k}} h_{j,k}(x)\,dx = 0,$$

and

$$\int_{\mathbf{R}} |h_{j,k}(x)|^2\,dx = \int_{I_{j,k}} |h_{j,k}(x)|^2\,dx = 1.$$

Orthogonality of functions.

Definition. We define the *inner product* of two functions defined on an interval I by

$$\langle f, g\rangle = \int_I f(x)\,\overline{g(x)}\,dx.$$

Two functions on I are said to be *orthogonal* if $\langle f, g\rangle = 0$.

A collection of functions $\{g_n(x)\}_{n\in\mathbf{N}}$ on I is an *orthogonal system* (on I) provided that

(a) $\langle g_n, g_m\rangle = 0$ if $n \neq m$, and

(b) $\langle g_n, g_n\rangle = \int_I |g_n(x)|^2\,dx > 0$. (This says in particular that none of the $g_n(x)$ can be identically zero.)

The collection $\{g_n(x)\}_{n\in\mathbf{N}}$ is an *orthonormal system* (on I) provided that it is an orthogonal system on I and

(b') $\langle g_n, g_n\rangle = \int_I |g_n(x)|^2\,dx = 1$.

Orthogonality of the Haar system.

Theorem. The Haar system (on $\mathbf{R}$) is an orthonormal system on $\mathbf{R}$.

Proof:

Theorem. Given any $j \in \mathbf{Z}$, the collection of scale j Haar scaling functions,

$$\{p_{j,k}(x) \colon k \in \mathbf{Z}\},$$

is an orthonormal system on $\mathbf{R}$.

Theorem. Given $J \in \mathbf{Z}$, the following hold.

(a) $\{p_{J,k}(x),\, h_{j,k}(x) \colon j \geq J,\, k \in \mathbf{Z}\}$ is an orthonormal system (on $\mathbf{R}$).

(b) $\{p_{J,k}(x),\, h_{J,k}(x) \colon k \in \mathbf{Z}\}$ is an orthonormal system (on $\mathbf{R}$).

The Splitting Lemma.

Lemma. Let $j \in \mathbf{Z}$, and let $g_j(x)$ be a scale j dyadic step function. Then $g_j(x)$ can be written as $g_j(x) = r_{j-1}(x) + g_{j-1}(x)$, where $r_{j-1}(x)$ has the form

$$r_{j-1}(x) = \sum_k a_{j-1}(k)\, h_{j-1,k}(x),$$

for some coefficients $\{a_{j-1}(k)\}_{k \in \mathbf{Z}}$, and $g_{j-1}(x)$ is a scale $j-1$ dyadic step function.

Moreover, $g_{j-1}(x)$ and $r_{j-1}(x)$ are orthogonal.

Proof:

The Haar basis on $[0,1]$.

Definition. For any integer $J \geq 0$, the *scale J Haar system on* $[0,1]$ is the collection

$$\{p_{J,k}(x) \colon 0 \leq k \leq 2^J - 1\} \\ \cup \{h_{j,k}(x) \colon j \geq J;\ 0 \leq k \leq 2^j - 1\}.$$

When $J = 0$, this collection will be referred to simply as the *Haar system on* $[0,1]$.

Theorem. The Haar system on $[0,1]$ is a complete orthonormal system on $[0,1]$. This means that every function $f(x)$ defined on $[0,1]$ can be written as a convergent *Fourier series* as

$$f(x) = \langle f, p_{0,0}\rangle\, p_{0,0}(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \langle f, h_{j,k}\rangle\, h_{j,k}(x)$$

where the numbers $\{\langle f, p_{0,0}\rangle, \langle f, h_{j,k}\rangle\}$ are referred to as the *Haar coefficients* of f. The convergence of the series is in the sense of L^2 or *mean square* convergence.

Proof:

(1) That the Haar system is an orthonormal system has already been established.

(2) It remains to show that each function $f(x)$ on $[0, 1]$ has the representation described. This will follow directly from the Splitting Lemma once we establish the following claim. The claim will not be proved here but I will argue that it is plausible.

Claim. We can find a sequence of functions $\{h_j\}_{j=0}^{\infty}$ such that each h_j is a scale j dyadic step function, and that $\lim_{j\to\infty} h_j = f$ in the sense of convergence in mean square (or L^2). In other words, $\int_0^1 |f(x) - h_j(x)|^2\,dx \to 0$ as $j \to \infty$.

(3) Assuming the claim is true, let $e_j(x) = f(x)-h_j(x)$. Then $\int_0^1 |e_j(x)|^2\,dx \to 0$ as $j \to \infty$.

(4) Fix any $J \geq 0$ and take $h_J(x)$ from the above sequence. We will have h_J play the role of g_J in the Splitting Lemma. By the Splitting Lemma, we can write

$$\begin{aligned}
h_J(x) &= g_{J-1}(x) + \sum_{k=0}^{2^{J-1}-1} a_{J-1}(k)\, h_{J-1,k}(x) \\
&= g_{J-2}(x) + \sum_{j=J-2}^{J-1} \sum_{k=0}^{2^j-1} a_j(k)\, h_{j,k}(x) \\
&= g_{J-3}(x) + \sum_{j=J-3}^{J-1} \sum_{k=0}^{2^j-1} a_j(k)\, h_{j,k}(x) \\
&= \cdots \\
&= g_0(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} a_j(k)\, h_{j,k}(x).
\end{aligned}$$

(5) Since $g_0(x)$ is a scale 0 dyadic step function, it is a constant multiple of $p_{0,0}(x)$. Hence we have established that for some constants $b_0(0)$ and $a_j(k)$ and for each J,

$$\begin{aligned} f(x) &= b_0(0)\, p_{0,0}(x) \\ &\quad + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} a_j(k)\, h_{j,k}(x) + e_J(x). \end{aligned}$$

(6) By general properties of orthonormal systems that we will learn later, we can specify the coefficients above to be just what we want them to be. Specifically, we can write

$$\begin{aligned} f(x) &= \langle f, p_{0,0} \rangle\, p_{0,0}(x) \\ &\quad + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, h_{j,k} \rangle\, h_{j,k}(x) + \widetilde{e}_J(x) \end{aligned}$$

where $\int_0^1 |\widetilde{e}_J(x)|^2\, dx \le \int_0^1 |e_J(x)|^2\, dx$.

As $J \to \infty$, $\int_0^1 |\widetilde{e}_J(x)|^2 \to 0$ so that

$$f(x) = \langle f, p_{0,0} \rangle\, p_{0,0}(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \langle f, h_{j,k} \rangle\, h_{j,k}(x).$$