Section 12.1, Exercise 4.

(b). By a result in the book we already know that $\text{Vol}(E_1 \cup E_2) \leq \text{Vol}(E_1) + \text{Vol}(E_2)$ always. So we must show that when $\text{Vol}(E_1 \cap E_2) = 0$ then $\text{Vol}(E_1 \cup E_2) \geq \text{Vol}(E_1) + \text{Vol}(E_2)$. We will also use the topological facts that $E_1 \cup E_2 = E_1 \cup E_2$ and that $E_1 \cap E_2 = E_1 \cap E_2$.

So let $\epsilon > 0$ and choose a grid $G$ such that $V(E_1 \cap E_2; G) < \epsilon$. Also note that a rectangle $R \in G$ satisfies $R \cap E_1 \neq \emptyset$ and $R \cap E_2 \neq \emptyset$ if and only if $R \cap (\overline{E_1} \cap \overline{E_2}) \neq \emptyset$.

Therefore

$$V(E_1 \cup E_2; G) = \sum_{R \cap E_1 \neq \emptyset} |R| + \sum_{R \cap E_2 \neq \emptyset} |R| - \sum_{R \cap (\overline{E_1} \cap \overline{E_2}) \neq \emptyset} |R|$$

$$= V(E_1; G) + V(E_2; G) - V(E_1 \cap E_2; G)$$

$$\geq V(E_1; G) + V(E_2; G) - \epsilon$$

$$\geq \text{Vol}(E_1) + \text{Vol}(E_2) - \epsilon$$

Taking the infimum of the left side over all such grids $G$ gives

$$\text{Vol}(E_1 \cup E_2) \geq \text{Vol}(E_1) + \text{Vol}(E_2) - \epsilon$$

Since $\epsilon > 0$ was arbitrary, we have

$$\text{Vol}(E_1 \cup E_2) \geq \text{Vol}(E_1) + \text{Vol}(E_2)$$

as required.

Exercise 5.

(a). Since $E^\circ \subseteq E \subseteq \overline{E}$ it follows that $\overline{E} \setminus E \subseteq \overline{E} \setminus E^\circ = \partial E$ and that $E \setminus E^\circ \subseteq \overline{E} \setminus E^\circ = \partial E$. Since $E$ is a Jordan region, $\text{Vol}(\partial E) = 0$. Hence both sets $\overline{E} \setminus E$ and $E \setminus E^\circ$ are subsets of sets with volume zero, so both are Jordan regions with volume zero. Therefore $E^\circ$ and $E$ differ from $E$ by sets of volume zero, hence each is a Jordan region (there is a remark in the section that asserts both of the above named facts).

(b). Since $E^\circ \subseteq E \subseteq \overline{E}$ it follows from one of the results in the book that $\text{Vol}(E^\circ) \leq \text{Vol}(E) \leq \text{Vol}(\overline{E})$. But $\overline{E} = E^\circ \cup (\overline{E} \setminus E^\circ)$ Hence again citing a result from the book,

$$\text{Vol}(\overline{E}) \leq \text{Vol}(E^\circ) + \text{Vol}(\overline{E} \setminus E^\circ) = \text{Vol}(E^\circ).$$

Therefore

$$\text{Vol}(E^\circ) = \text{Vol}(E) = \text{Vol}(\overline{E})$$

as required.
Section 12.4, Exercise 4.

(a). Let $E$ be the unit ball in $\mathbb{R}^3$ given by $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$, and consider the transformation $\varphi(x, y, z) = (ax, by, cz)$. Then $\varphi(E)$ is the ellipsoid given in the problem because if $(u, v, w) \in \varphi(E)$ then $u = ax$, $v = by$, and $w = cz$ for some $(x, y, z) \in E$. Hence

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = x^2 + y^2 + z^2 = 1.$$ 

Also it is easy to see that $\Delta \varphi(x, y, z) = abc$. Therefore by the change of variables formula

$$Vol(\varphi(E)) = \int_{\varphi(E)} 1 \, dV = \int_E \left| \Delta \varphi(x, y, z) \right| dV = abc \int_E 1 \, dV = \frac{4}{3} \pi abc$$

as required.

Exercise 6.

By the change of variables formula

$$Vol(f(R_r(x_0))) = \int_{f(B_r(x_0))} 1 \, dV = \int_{B_r(x_0)} |\Delta f(x)| \, dx.$$ 

Therefore,

$$\frac{Vol(f(R_r(x_0)))}{Vol(B_r(x_0))} = \frac{1}{Vol(B_r(x_0))} \int_{B_r(x_0)} |\Delta f(x)| \, dx.$$ 

But by assumption, $\Delta f(x)$ is continuous on $B_r(x_0)$ for all $r > 0$ sufficiently small. Hence the hypotheses in the result referred to in the hint to this problem (Problem 5, p. 406 I believe) are satisfied and we can conclude that

$$\lim_{r \to 0} \frac{1}{Vol(B_r(x_0))} \int_{B_r(x_0)} |\Delta f(x)| \, dx = |\Delta f(x_0)|$$