Find the solution to the initial value problem.

\[ y''' - 4y'' + 4y' = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 0. \]

**Solution:** The characteristic equation of this ODE is \( r^3 - 4r^2 + 4r = 0 \), which factors as \( r(r - 2)^2 = 0 \) and hence has solutions \( r_1 = 0, \; r_2 = 2, \) where \( r_2 \) is a double root. Therefore the general solution is given by

\[ y(t) = c_1 + c_2 e^{2t} + c_3 t e^{2t}. \]

We have then \( y'(t) = 2c_2 e^{2t} + c_3 (2t e^{2t} + e^{2t}) \) and \( y''(t) = 4c_2 e^{2t} + c_3 (4t e^{2t} + 4e^{2t}) \). Plugging in the initial conditions gives the system of equations

\[
\begin{align*}
1 & = c_1 + c_2 \\
-1 & = 2c_2 + c_3 \\
0 & = 4c_2 + 4c_3.
\end{align*}
\]

Subtracting twice the second equation to the third gives \( c_3 = 1 \), and plugging this into the last equation gives \( c_2 = -1 \), and plugging this into the first equation gives \( c_1 = 2 \). Hence the final solution is

\[ y(t) = 2 - e^{2t} + t e^{2t}. \]

Find the solution to the initial value problem.

\[ y''' + y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 2. \]

**Solution:** The characteristic equation of this ODE is \( r^3 + r = 0 \), which factors as \( r(r^2 + 1) = 0 \) and hence has solutions \( r_1 = 0, \; r_2 = i, \; r_3 = -i \). Therefore the general solution is given by

\[ y(t) = c_1 + c_2 \cos(t) + c_3 \sin(t). \]

We have then \( y'(t) = -c_2 \sin(t) + c_3 \cos(t) \) and \( y''(t) = -c_2 \cos(t) - c_3 \sin(t) \). Plugging in the initial conditions gives the system of equations

\[
\begin{align*}
0 & = c_1 + c_2 \\
1 & = c_3 \\
2 & = -c_2
\end{align*}
\]
which leads to \( c_1 = 2, \ c_2 = -2, \) and \( c_3 = 1. \) Hence the final solution is
\[ y(t) = 2 - 2 \cos(t) + \sin(t). \]

---

Find the solution to the initial value problem.
\[ y''' + 5y'' + 6y' = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1. \]

Solution: The characteristic equation of this ODE is \( r^3 + 5r^2 + 6r = 0, \) which factors as \( r(r + 3)(r + 2) = 0 \) and hence has solutions \( r_1 = 0, \ r_2 = -3, \ r_3 = -2. \) Therefore the general solution is given by
\[ y(t) = c_1 + c_2 e^{-3t} + c_3 e^{-2t}. \]

We have then \( y'(t) = -3c_2 e^{-3t} - 2c_3 e^{-2t} \) and \( y''(t) = 9c_2 e^{-3t} + 4c_3 e^{-2t}. \) Plugging in the initial conditions gives the system of equations
\[
\begin{align*}
0 & = c_1 + c_2 + c_3 \\
0 & = -3c_2 - 2c_3 \\
1 & = 9c_2 + 4c_3.
\end{align*}
\]

Adding 3 times the second equation to the third gives \( c_3 = -1/2, \) and plugging this into the second equation gives \( c_2 = 1/3, \) and plugging this into the first equation gives \( c_1 = 1/6. \) Hence the final solution is
\[ y(t) = \frac{1}{6} + \frac{1}{3} e^{-3t} - \frac{1}{2} e^{-2t}. \]