MATH 214 – EXAM 2 VERSION 1 – SOLUTIONS

1. (10 pts.) Find the solution of the initial value problem
\[ y'' - 2y' + y = t e^{2t}, \quad y(0) = 1, \quad y'(0) = 0. \]

Solution:

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is \( r^2 - 2r + 1 = 0 \) which factors to \( (r - 1)^2 = 0 \) giving a solution of \( r = 1 \) of multiplicity 2. Hence the general solution is
\[ y(t) = c_1 e^t + c_2 t e^t. \]

We will use the method of undetermined coefficients to find a particular solution, of the form \( Y(t) = A t e^{2t} + B e^{2t} \). This gives \( Y'(t) = 2A t e^{2t} + A e^{2t} + 2B e^{2t} \) and \( Y''(t) = 4A t e^{2t} + 4A e^{2t} + 4B e^{2t} \). Plugging into the original equation gives
\[ 4A t e^{2t} + 4A e^{2t} + 4B e^{2t} - 4A t e^{2t} - 2A e^{2t} - 4B e^{2t} + A t e^{2t} + B e^{2t} = t e^{2t}. \]

Gathering like terms, this leads to
\[ A t e^{2t} + (2A + B) e^{2t} = t e^{2t}. \]

Which solves to \( A = 1, \) \( B = -2. \) Hence the general solution is
\[ y(t) = c_1 e^t + c_2 t e^t + t e^{2t} - 2 e^{2t}. \]

Plugging in the initial conditions leads to the system
\[
\begin{align*}
1 &= c_1 - 2 \\
0 &= c_1 + c_2 + 1 - 4
\end{align*}
\]

yielding a solution of \( c_1 = 3 \) and \( c_2 = 0. \) Hence the final solution is
\[ y(t) = 3 e^t + t e^{2t} - 2 e^{2t}. \]

2. (10 pts.) Use the method of variation of parameters to find a particular solution to the differential equation
\[ y'' - y' - 2y = 2 e^{-t}. \]

Solution:
First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^2 - r - 2 = 0$ which factors to $(r+1)(r-2) = 0$ giving solutions of $r = -1$ and $r = 2$. Hence the general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{2t}.$$  

We will use the method of variation of parameters to find a particular solution, of the form $Y(t) = u_1(t) e^{-t} + u_2(t) e^{2t}$. The variation of parameters formula gives the system

$$u_1'(t) e^{-t} + u_2'(t) e^{2t} = 0$$

$$-u_1'(t) e^{-t} + 2u_2'(t) e^{2t} = 2 e^{-t}$$

Adding the equations gives $3u_2'(t) e^{2t} = 2 e^{-t}$ which solve to $u_2'(t) = (2/3)e^{-3t}$ or $u_2(t) = -(2/9)e^{-3t}$. Plugging into the first equation gives $u_1'(t) = -2/3$ or $u_1 = -(2/3)t$. Hence a particular solution is given by

$$Y(t) = -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t}.$$  

The second term in that sum could be omitted since it is a solution to the homogeneous equation.

3. (10 pts.) Find the general solution to the differential equation

$$y'' - y' = 6t.$$  

\textbf{Solution:}

First we need to find the general solution to the homogeneous system. The characteristic equation for this ODE is $r^3 - r^2 = 0$ which factors to $r(r - 1)(r - 1) = 0$ giving solutions of $r = 0$ of multiplicity 2 and $r = 1$. Hence the general solution is

$$y(t) = c_1 + c_2 t + c_3 e^t + Y(t)$$

where $Y(t)$ is a particular solution. We will find $Y(t)$ by the method of undetermined coefficients. If $Y(t) = At^3 + Bt^2$ then $Y'(t) = 3At^2 + 2Bt$, $Y''(t) = 6At + 2B$, and $Y'''(t) = 6A$. Plugging into the original equation gives

$$6A - 6At + 2B = 6t$$

which gives $A = -1$ and $B = 3$. Hence the general solution to the equation is

$$y(t) = c_1 + c_2 t + c_3 e^t - t^3 + 3t^2.$$  

4. (10 pts.) Find the general solution to the differential equation

$$y^{(4)} - 5y'' + 4y = 0.$$  

\textbf{Solution:}
The characteristic equation for this ODE is \( r^4 - 5r^2 + 4 = 0 \) which factors to \((r^2 - 1)(r^2 - 4) = 0\) and again to \((r - 1)(r + 1)(r - 2)(r + 2) = 0\) giving solutions of \( r = 1, -1, 2, -2 \). Hence the general solution is

\[
y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}.\]