Quiz 3 on Thursday will be on 2.1 and 2.2.

2.4 Linear vs Nonlinear ODEs
(Existence and Uniqueness)

Looking at \( \frac{dy}{dt} = f(t,y) \) \( y(t_0) = y_0 \) (IVP)

1. Can we guarantee that a solution exists?
2. Can we guarantee that there is only one solution?

Remember this is an IVP.

In all examples, we have seen there is only one solution through each point \((t_0, y_0)\).

Can this happen?
know that the slope here must be \( f(t_0, y_0) \)

2 solutions thru same point?
3. Can we always find the interval of \( t \) where the solution (or solutions) exists? (We have seen examples where the solution exists for some \( t \) but not all \( t \).

\[ t + y' + 2y = 4t^2 \quad y(1) = 2 \]

Already solved this one:

\[ y' + \frac{2}{t} y = 4t \]

\[ M = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2 \]

\[ \frac{d}{dt} (t^2 y) = 4t \cdot t^2 = 4t^3 \]

\[ t^2 y = t^4 + c \]

\[ y = t^2 + \frac{c}{t^2} \]

\[ y(1) = 2 \rightarrow 2 = 1 + c \]

\[ \rightarrow c = 1. \]

\[ y \]

\[ t \]

\[ \text{Solution only exists for } t > 0. \]

\[ \text{Interval on which solutions exist depends on the equation and on the initial condition.} \]
Return to the uniqueness question.

\[ \frac{dy}{dt} = y^{1/3} \quad y(0) = 0 \]

Without doing any work we see that \( y(t) = 0 \) is a solution.

\[ \frac{dy}{y^{1/3}} = dt \]

\[ \frac{3}{2} y^{2/3} = t + c \]

\[ y(0) = 0 \]

\[ c = 0 \]

\[ \frac{3}{2} y^{2/3} = t \]

\[ y = \left( \frac{2}{3} t \right)^{3/2} \]

Valid for \( t \geq 0 \) on \( y \).

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So what is the bottom line?

**Theorem 2.4.2** \( \frac{dy}{dt} = f(t,y) \) \( y(t_0) = y_0 \)

If \( f, \frac{df}{dy} \) continuous in some rectangle \( x < t < \beta, \gamma < y \leq \delta \) then there is unique solution on some interval within \( (x, \beta) \).

\[ y \]

\[ (t_0, y_0) \]

\[ t \]
Think about \( y_1 = y^{1/3} \), \( y(0) = 0 \).

\[ f(t, y) = y^{1/3}, \quad \frac{dy}{dt} = \frac{1}{3} y^{-2/3} \text{ not continuous in any interval containing } y = 0. \]

Suppose instead we take \( y(0) = 1 \).

Then we are OK.

\[ \frac{3}{2} y^{2/3} = t + c \quad y(0) = 1 \]

\[ \frac{3}{2} = c \]

\[ \frac{3}{2} y^{2/3} = t + \frac{3}{2} \]

\[ y = \left( \frac{2}{3} t + 1 \right)^{3/2} \]

Valid for \( t > -\frac{3}{2} \).

\[ \int \text{ graph here.} \]

\[ \text{One solution only (0, 1)} \]
For linear equations we have:

\[
\frac{dy}{dt} + p(t)y = g(t) \quad y(t_0) = y_0
\]

\[
\frac{dy}{dt} = -p(t)y + g(t) \quad \frac{df}{dy} = -p(t)
\]

\[f(t, y) \quad \text{so if } p(t) \text{ is continuous in an interval around } t_0, \text{ we have a unique solution. Also } g(t) \text{ needs to be continuous.}
\]

\[e.g., \#2 \]

\[
t(t-u)y' + y = 0 \implies y' + \frac{1}{t(t-4)}y = 0
\]

\[y(2) = 1 \quad p(t) \text{ is continuous on } (y \text{ on } (-\infty, 0) \cup (0, 4) \cup (4, \infty)).
\]

Since \(t_0 = 2 \in (0, 4) \) \( \implies \) can guarantee a solution on \((0, 4)\).

\[e.g. \text{ if } y(2) = 0 \text{ then I get a solution for all } t.
\]

\[y(2) = 0 \text{ then the solution is } y(t) = 0.
\]
5. \((1-t^2)y' + 2ty = 3t^2\) \(y(1) = -3\)

\[
y' + \frac{2t}{1-t^2} y = \frac{3t^2}{1-t^2}
\]

Unique solutions on the intervals: 
\((-\infty, -2), (-2, 2), (2, \infty)\).

Since \(t_0 = 1\), I can guarantee a unique solution on \((-2, 2)\).

14. \(y' = 2ty^2\) \(y(0) = y_0\).

nonlinear

\[
\frac{dy}{y^2} = 2t\ dt
\]

\[-\frac{1}{y} = t^2 + C\]

\[
y = \frac{-1}{t^2 + C}
\]

\[
y(0) = y_0
\]

\[
y_0 = \frac{-1}{C} \Rightarrow C = -\frac{1}{y_0}
\]

If \(y_0 \leq 0\) solution exists for all \(t\).

If \(y_0 > 0\) then solution exists only for \(|t| < \frac{1}{\sqrt{y_0}}\).
2.6 Exact Equations + Integrating Factors.

Exact differential

Vector field: \( \mathbf{F}(x, y) = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} \)

Q: Is \( \mathbf{F} \) a gradient field?

That is, \( \mathbf{F} = \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \)

Is there a function \( f(x, y) \) such that \( \frac{\partial f}{\partial x} = M(x, y) \) and \( \frac{\partial f}{\partial y} = N(x, y) \)?

If so, we say \( M(x, y) \, dx + N(x, y) \, dy \) is an exact differential.

Exact equation has the form

\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0 \]

In general these are not linear and not separable.

These can be solved if \( M \, dx + N \, dy \) is an exact differential.

Suppose that \( M = \frac{\partial \psi}{\partial x}, \ N = \frac{\partial \psi}{\partial y} \). Then eqn.

becomes \( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0 \)

\[ \frac{d}{dx} \left( \psi(x, y(x)) \right) = 0 \]

\( \psi(x, y) = c \). Solution (implicit solution)
\( (2x + y^2) + (2xy)y' = 0 \)

\[ M = \frac{\partial y}{\partial x}, \quad N = \frac{\partial y}{\partial y} \]

Look for \( y(x, y) \) s.t. \( M = \frac{\partial y}{\partial x}, \quad N = \frac{\partial y}{\partial y} \)

\[
\frac{dy}{dx} = 2x + y^2 \quad \frac{dy}{dy} = 2xy
\]

\[ y = x^2 + xy^2 + g(y) \]

\[ 2xy + g'(y) = 2xy \]

\[ g'(y) = 0 \quad g = \text{const.} \]

I can take \( g = 0 \) since I only need a solution, not all solutions.

Solution: \( y(x, y) = x^2 + xy^2 \)

Solution to ODE is \( x^2 + xy^2 = c \)

or \( y^2 = \frac{c - x^2}{x} \)

\[ y = \pm \left( \frac{c - x^2}{x} \right)^{1/2} \]

\text{implicit form}

\text{explicit form}
\[ (y \cos x + 2x e^y) + (\sin x + x^2 e^y - 1)y' = 0 \]

Can I tell without actually solving, that this is exact? Yes.

\[
\begin{align*}
M &= \frac{\partial M}{\partial x} \\
N &= \frac{\partial N}{\partial y} \\
M_y &= \frac{\partial^2 y}{\partial y \partial x} \\
N_x &= \frac{\partial^2 y}{\partial x \partial y}
\end{align*}
\]

Thm: \( M \, dx + N \, dy \) is exact differential if \( M_y = N_x \).

\[
\begin{align*}
M_y &= \cos x + 2x e^y \\
N_x &= \cos x + 2x e^y
\end{align*}
\]

\[
\begin{align*}
\frac{\partial M}{\partial x} &= y \cos x + 2x e^y \\
y &= y \sin x + x^2 e^y + g(y) \\
\frac{\partial y}{\partial y} &= \sin x + x^2 e^y + g'(y) = \sin x + x^2 e^y - 1 \\
g'(y) &= -1 \quad \text{so} \quad g(y) = -y
\end{align*}
\]

\[
\therefore \quad y = y \sin x + x^2 e^y - y \quad \text{solution is:} \quad y \sin x + x^2 e^y - y = c.
\]