Laplace Transforms

Forcing functions
- second order linear equations w/const coeff&
  model many kinds of oscillatory motion
- mass + spring systems,
- electrical circuits

Mass + Spring systems
\[ m \dddot{u} + \ddot{u} + ku = F \]

A. Unforced systems,

1. Undamped \( \delta = 0 \).
   oscillatory motion \( u(t) = R \cos(\omega_0 t - \theta) \)
   \( \omega_0 \) - natural frequency \( \omega_0 = \sqrt{\frac{k}{m}} \)
   \( R, \theta \) depend on initial conditions.

2. Damping \( \delta > 0 \).
   a. Small damping \( 0 < \delta < 2\sqrt{\frac{k}{m}} \)
damped oscillations
   b. Critical or overdamped \( \delta \geq 2\sqrt{\frac{k}{m}} \)
decaying solutions - no oscillation
B. Forced systems \( F \neq 0 \)

1. Undamped \( \delta = 0 \)

   a. \( \omega = \omega_0 \rightarrow \text{resonance} \) and \( u(t) \rightarrow \infty \) as \( t \rightarrow \infty \).

   b. \( \omega \neq \omega_0 \rightarrow \text{beat} \)

2. Damped \( \delta > 0 \)

   \[ m u'' + \delta u' + ku = F_0 \cos \omega t \quad \omega \neq \omega_0 = \sqrt{\frac{F_0}{m}} \]

   \[ u(t) = R_1 e^{-\delta t/2m} \cos (\omega t - S_1) \quad R_1, S_1 \text{ depend on initial cond.} \]

   \[ \uparrow \text{quasi-frequency,} \]

   \[ + (A \cos \omega t + B \sin \omega t) \]

   (transient solution)

   dies out as \( t \rightarrow \infty \)

   \[ \begin{align*}
   R_2 \cos (\omega t - S_2) & \leftarrow \text{persists} \\
   \uparrow \text{How does } R_2 \text{ depend on } \omega \text{ and } F_0? \end{align*} \]

   (steady-state solution)
After some work:

\[ R_2 = \left( \frac{E_0}{k} \right) \left[ \frac{1}{(1 - \left( \frac{w}{w_0} \right)^2)^2 + \left( \frac{\gamma^2}{\omega_0^2} \right) \left( \frac{w}{w_0} \right)^2} \right]^{1/2} \]

If \( \frac{w}{w_0} \) is very small then \([ - ] \times 1\)
and \( R_2 \approx \frac{E_0}{k} \cdot \frac{w}{w_0} \) small means forcing has low frequency (slow oscillations).

If \( \frac{w}{w_0} \approx 1 \) then \( R_2 \approx \frac{E_0}{k} \cdot \frac{\sqrt{\gamma k}}{\omega_0} \)
so oscillations are large if \( \gamma \) is small i.e. small damping.

If \( \frac{w}{w_0} \) very large then forcing has very fast oscillations and \([ - ] \) is very small so \( R_2 \) is very small.
6.4 Discontinuous Forcing Functions

We have looked at periodic forcing. What would discontinuous forcing look like?

\[ mu'' + ku + bu = g(t) \]

\[ u(0) = 1, \quad u'(0) = 0 \]

Discontinuous forcing is like suddenly increasing gravity (or changing the mass, or turning on floor magnet, etc.)

1) Oscillatory motion for \( 0 \leq t < c \)

Solving \( mu'' + ku = 0 \), \( u(0) = 1 \), \( u'(0) = 0 \)

\[ y_A(t) = R \cos (w_0 t - \delta) = \cos (w_0 t) \]

\[ R = 1 \]

\[ \delta = 0 \]

2) at \( t = c \) we are solving

\( mu'' + ku = 1 \) with new initial conditions

\[ u(c) = y_A(c) \quad u'(c) = y_A'(c) \]
What will solution look like?

\[ y_{B}(t) = R_1 \cos(\omega t - \delta) + \frac{1}{k} \]

\[ \text{homogeneous solution} \]
\[ \text{particular solution} \]

What happens at \( t = c \)?
- A jump? NO
- A corner? NO
- A discontinuous second derivative? YES.

**Electrical circuit**

\[ LQ'' + RQ' + \frac{1}{C}Q = E \]

- \( L \)-inductor
- \( C \)-capacitor
\[ E(t) : \quad \begin{array}{c}
\frac{d^2 y}{dt^2} + y' + 2y = u_5(t) - u_{20}(t) = 9(t) \\
0 \quad 5 \quad 20
\end{array} \]

\[ y(0) = 0, \quad y'(0) = 0 \]

This is like flipping on a switch.

Other examples:

Impulsive force:

Ramp loading:

\[ \begin{array}{c}
\frac{d^2 y}{dt^2} + y' + 2y = u_5(t) - u_{20}(t) = 9(t) \\
0 \quad 5 \quad 20
\end{array} \]
\[ 2 \mathfrak{L} y' + \mathfrak{L} y^2 + 2 \mathfrak{L} y^3 = \mathfrak{L} u_1 - \mathfrak{L} u_2 \]
\[ 2 \left( s^2 \mathfrak{L} y - s y(0) - y(0) \right) + s \mathfrak{L} y^2 - y(0) + 2 \mathfrak{L} y^3 \]
\[ = \mathfrak{L} u_1 - \mathfrak{L} u_2 \]
\[ (2s^2+s+2) \mathfrak{L} y = \frac{1}{s} (e^{-5s} - e^{-20s}) \]
\[ \mathfrak{L} y^3 = \frac{e^{-5s} - e^{-20s}}{s (2s^2+s+2)} \]

\[
\begin{bmatrix}
1 \\
\frac{1}{s (2s^2+s+2)} \\
\end{bmatrix} = \frac{A}{s} + \frac{B s + C}{2s^2+s+2} \]

\[
= \frac{A (2s^2+s+2) + s (B s + C)}{s (2s^2+s+2)} \]

\[ A (2s^2+s+2) + s (B s + C) = 1 \quad s = 0 \]

\[ 2A = 1 \rightarrow A = \frac{1}{2} \]

\[ \frac{d}{ds} \left( A (4s+1) + 2Bs + C \right) = 0 \quad s = 0 \]

\[ A + C = 0 \rightarrow C = -\frac{1}{2} \]

\[ \frac{d^2}{ds^2} \left( 4A + 2B = 0 \right) \rightarrow B = -\frac{1}{2} \]
\[ f(t) = (e^{-5s} - e^{-20s}) \left( \frac{1}{2} s - \frac{s+ \frac{1}{2}}{2s^2 + \frac{1}{2} s + 1} \right) \]

\[ = \frac{1}{2} (e^{-5s} - e^{-20s}) \left( \frac{s}{s} - \frac{s+ \frac{1}{4}}{(s+ \frac{1}{4})^2 + \frac{15}{16}} \right) \]

\[ = \frac{1}{2} (e^{-5s} - e^{-20s}) \left( \frac{1}{s} - \frac{s+ \frac{1}{4}}{(s+ \frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{4} \frac{1}{(s+ \frac{1}{4})^2 + \frac{15}{16}} \right) \]

\[ g^{-1}(s) = 1 - e^{-\frac{t}{4}} \cos \left( \frac{\sqrt{15}}{4} t \right) \]

\[ - \frac{1}{4} \frac{\sqrt{15}}{\sqrt{15}} e^{-\frac{t}{4}} \sin \left( \frac{\sqrt{15}}{4} t \right) \]

\[ = h(t) \]

\[ L_y(t) = \frac{1}{2} \left( e^{-5s} - e^{-20s} \right) h(t) \]