looked at \( \frac{dy}{dt} = f(t, y) \)

\[
\frac{dy}{dt} = Ay + B, \quad A, B \text{ constants.}
\]

Falling object \quad Mice + Owls

\[
\frac{dv}{dt} = g - \frac{v}{m} \quad \frac{dp}{dt} = vp - b
\]

B. \[
\frac{dv}{dt} = 9.8 - 0.2v = \frac{49 - v}{5}
\]

\[
\frac{dv}{49-v} = \frac{1}{5} dt
\]

\[
v = 49 - c e^{-\frac{1}{5}t}
\]

\[
v(0) = 49 - c \quad c = 49 - v(0)
\]

\[
v(0) > 49 \quad c = 49 - v(0) < 0
\]

\[
v(0) < 49 \quad c > 0
\]

E.g. \( v(0) = 0 \) \(
\quad \text{object is alway up.}
\)

\[
c = 49 \quad v(t) = 49 - 49 e^{-\frac{1}{5}t}
\]

\[
= 49(1 - e^{-\frac{1}{5}t})
\]
If an object is dropped from 300 m when does it hit the ground?

\[ u(t) = \frac{dx}{dt}, \quad x = \text{distance the object has fallen.} \]

\[ \frac{dx}{dt} = 49 (1 - e^{-\frac{1}{5}t}) \]

\[ x(t) = 49 (t + 5 e^{-\frac{1}{5}t}) + c \]

Find c.

\[ x(0) = 0 \]

\[ 0 = 49 (5) + c \]

\[ c = -245 \]

\[ x(t) = 49 (t + 5 e^{-\frac{1}{5}t}) - 245 \]

Solve:

\[ 49 (t + 5 e^{-\frac{1}{5}t}) - 245 = 300 \]

\[ 49 (t + 5 e^{-\frac{1}{5}t}) = 545 \]

\[ t + 5 e^{-\frac{1}{5}t} = \frac{545}{49} \approx 11.1 \]

\[ t \approx 10.5 \text{ sec.} \]

How fast at impact?

\[ u(10.5) = 49 (1 - e^{-\frac{1}{5}(10.5)}) \approx 43.0 \text{ m/sec} \]
#2) \( a \) \( \frac{dy}{dt} = y - 5 \quad y(0) = y_0 \)

\[
\frac{dy}{y-5} = dt \quad \rightarrow \quad \ln|y-5| = t + c
\]

\[
|y-5| = e^c e^t
\]

\[
y-5 = c e^t
\]

\[
y = 5 + c e^t
\]

\[
y(0) = 5 + c - \Rightarrow y_0
\]

\[
c = y_0 - 5
\]

\[
y = 5 + (y_0 - 5) e^t
\]

\[
y_0 = 4 \quad y = 5 - e^t
\]

\[
y_0 = 6 \quad y = 5 + e^t
\]
#9) (a) \( q = \) amount of dye in pool in kg.
\( t = \) time in minutes.

Concentration of dye: \( \frac{8(t)}{60000} \) kg gal

How much dye is removed by filter?
\[
200 \frac{\text{gal}}{\text{min}} \cdot \frac{8(t)}{60000} \frac{\text{kg}}{\text{gal}} = \frac{8(t)}{300} \frac{\text{kg}}{\text{min}}.
\]

\[
\frac{dq}{dt} = -\frac{8}{300} \quad \text{\( q(0) = 5 \)}
\]

(b) \( q(t) = c e^{-\frac{t}{300}} \)
\[
q(0) = 5 \quad \Rightarrow \quad q(t) = 5 e^{-\frac{t}{300}}
\]

(c) We need to have \( \frac{q}{60000} < 0.0002 \)

or \( q < 1.2 \) kg. \( 4h = 240 \) min.

\[
q(240) = 5 e^{-\frac{240}{300}} = 5 e^{-0.8} \approx 2.25
\]

(d) Solve \( q(t) = 1.2 \) i.e. \( 5 e^{-\frac{t}{300}} = 1.2 \)

(e) IUP becomes \( \frac{dq}{dt} = -\frac{E}{60000} q ; q(0) = 5 \)
1.3 Classification of DEs.

A. ODE vs. PDE

- Ordinary Partial

Ordinary DE involves \( y, y', y'', \ldots \)

So \( y = y(t) \) (one variable)

Partial DE involves \( \frac{dy}{dt}, \frac{dy}{dx}, \frac{dy}{dz}, \ldots \)

So \( y = y(t, x, z, w, \ldots) \)

B. Systems of ODE.

\[
\frac{dx}{dt} = f(t, x, y) \quad \text{solution is}
\]

\[
\frac{dy}{dt} = g(t, x, y) \quad \text{two functions}
\]

\[
x(t), y(t).
\]

E.g.

\[
\frac{dx}{dt} = ax + by + ct \quad a, b, c, d, e, f
\]

\[
\frac{dy}{dt} = dx + ey + ft \quad \text{constant}.
\]
C. Order of an ODE

order = highest derivative appearing in

the ODE or system of ODEs.

e.g. \( y''' + (y')^2 + e^y = 0 \) \( \text{ordu} = 3 \)

In general an \( n \text{th} \) order ODE looks like

\[ F(t, y, y', y'', \ldots, y^{(n)}) = 0 \]

\( \neq \) implicit form of

ODE

some function

df \( n+2 \) variables

Above example corresponds to:

\[ F(t, y, y', y'', y''') = 0 \]

\[ F(x_1, x_2, x_3, x_4, x_5) = x_5 + x_3^2 + \epsilon e^{x_1} x_2 \cdot \]

In this class we will assume we can

write:

\[ y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \]

\( \text{e.g.} \)

\[ y''' = -(y')^2 - e^y \]

explicit

form of

ODE.
D. Linear vs. Non-linear.

A linear $n^{th}$ order ODE looks like

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

Non-linear means you can't do this,

Eq: $y''' + (y')^2 + e^t y = 0$ 3rd order non-linear

$y''' + 2\tan(t)y' + e^t y = \tan^2 t$ 3rd order linear.

$$\frac{d^2 \theta}{dt^2} + \frac{9}{L} \sin \theta = 0$$ 2nd order linear.

$$\frac{d^2 \theta}{dt^2} + \frac{9}{L} \theta = 0$$ (with $\theta = y$ here) 2nd order non-linear.

$yy' + 2ty + \sec(t) = 0$ 1st order non-linear

$$\frac{dy}{dt} = f(t, y)$$ 1st order (explicit form) (maybe linear, maybe not)
#(8) \[ y''' - 3y'' + 2y' = 0 \]

If \( y = e^{rt} \)

\[
\begin{align*}
y' &= re^{rt} \\
y'' &= r^2 e^{rt} \\
y''' &= r^3 e^{rt}
\end{align*}
\]

\[
r^3 e^{rt} - 3r^2 e^{rt} + 2re^{rt} = 0
\]

\[
er^t (r^3 - 3r^2 + 2r) = 0
\]

\[
r^3 - 3r^2 + 2r = 0
\]

\[
r (r^2 - 3r + 2) = 0
\]

\[
r (r - 1)(r - 2) = 0
\]

\[
r = 0, \ r = 1, \ r = 2 \]

3rd order linear (constant coefficients) homogeneous equations like this have solutions of the form \( y = e^{rt} \) for certain values of \( r \).
2.1 Linear equations; Integrating factors

\[ \frac{dy}{dt} = f(t, y) \quad \text{1st order.} \]

Our examples: \( \frac{dv}{dt} = g - \frac{v}{m} \) falling object

\[ \frac{dp}{dt} = rp - b \quad \text{Mice owls.} \]

General form: \( \frac{dy}{dt} + ay = b. \quad a, b \ \text{const.} \)

We can use separability of variables.

\[ \frac{dy}{dt} = -ay + b = -a \left( y - \frac{b}{a} \right) \]

\[ \frac{dy}{y - \frac{b}{a}} = -a \ dt \quad \rightarrow \ln \left| y - \frac{b}{a} \right| = -ae^t + c \]

\[ \rightarrow \left| y - \frac{b}{a} \right| = ce^{-at} \]

\[ \rightarrow y - \frac{b}{a} = c e^{-at} \quad \rightarrow \quad y = \frac{b}{a} + ce^{-at} \]

Different method.

Idea: Integrating factor.

Suppose we replace \( y \) by \( \mu(t) y \)

Then \( \frac{d}{dt} (\mu y) = \mu(t) \frac{dy}{dt} + \mu'(t) y \)
This looks like LHS of \( \frac{dy}{dt} + ay = b \).

If \( \mu'(t) = a \mu(t) \) then I would have
\[
\frac{d}{dt} (\mu(t)y) = \mu(t) \frac{dy}{dt} + a \mu(t)y
\]
\[= \mu(t) \left( \frac{dy}{dt} + ay \right),\]
This would mean my equation becomes
\[
\frac{d}{dt} (\mu(t)y) = \mu(t) \left( \frac{dy}{dt} + ay \right) = \mu(t) b.
\]
If \( \mu'(t) = a \mu(t) \) then \( \mu(t) = e^{at} \)

Solve: \( \frac{dy}{dt} + ay = b \)
\[e^{at} \frac{dy}{dt} + a e^{at} y = e^{at} b \]
\[\frac{d}{dt} (e^{at} y) = e^{at} b,\]
\[e^{at} y = \frac{b}{a} e^{at} + c\]
\[y = \frac{b}{a} + c e^{-at}\] Same solution as before.