Vector functions

\[ \vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3 \]

set of 3-d vectors.

real numbers

\[ \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \]

sometimes

\[ \vec{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \]

graph of \( \vec{r}(t) \) is a curve.

But \( \vec{r}(t) \) gives a notion of motion along curve.

Operations on \( \vec{r}(t) \) are done component wise

a) magnitude  

b) vector addition  

c) scalar product  

de) cross product  

Also  

e) take limit  

f) differentiation  

g) integration

Differentiation

\[ y = f(x) \]

\[ f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \]

tangent line

secant line
Using $x$-y

\[ \frac{\vec{r}'(t)}{t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \]

has direction of second limit

direction of tangent line

Also write $\frac{d\vec{r}}{dt}$ for $\vec{r}'(t)$.

What is $|\vec{r}'(t)|$?

\[ |\vec{r}'(t)| = \lim_{\Delta t \to 0} \frac{|\vec{r}(t+\Delta t) - \vec{r}(t)|}{|\Delta t|} \]

= rate of change of displacement at $t$.

= speed of particle at $t$. 

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eq. 2) \[ \vec{r}(t) = (t^2 + 1) \hat{\imath} + (2t - 1) \hat{j} \quad t = \frac{1}{2} \]

a) Path of particle 

\[ \vec{r}(\frac{1}{2}) = \frac{5}{4} \hat{\imath} \hat{j}; \quad \vec{r}(0) = \hat{\imath} - \hat{j} \]

b) \[ \vec{r}'(t) = 2t \hat{\imath} + 2 \hat{j} \quad \vec{r}'(\frac{1}{2}) = \hat{\imath} + 2 \hat{j} \]

\[ \vec{r}''(t) = 2 \hat{\imath} \]

\[ |\vec{r}'(\frac{1}{2})| = (1 + 4)^{\frac{1}{2}} = \sqrt{5} \]

\[ |\vec{r}'(t)| = (4t^2 + 4)^{\frac{1}{2}} = 2(t^2 + 1)^{\frac{1}{2}} \]

\[ |\vec{r}''(t)| = (4)^{\frac{1}{2}} = 2 \]

\[ \vec{r}''''(t) = 2 \hat{\imath} \]

\[ |\vec{r}''''(t)| = (2)^{\frac{1}{2}} = \sqrt{2} \]

\[ \vec{r}''''(t) = \cos 2t \hat{\imath} + 3 \sin 2t \hat{j} ; \quad t = \theta \]

\[ (x, y) = (\cos 2t, 3 \sin 2t) \]

\[ \begin{align*}
\frac{y}{3} & = \sin 2t \\
x^2 + \left( \frac{y}{3} \right)^2 & = \cos^2 2t + \sin^2 2t = 1
\end{align*} \]
b) \( \vec{r}'(t) = -2 \sin t \hat{i} + 6 \cos 2t \hat{j} \)
\[ \vec{r}(0) = 6 \hat{j} \quad \vec{r}(0) = 2 \]

#10) \( \vec{r}(t) = (1+t) \hat{i} + \frac{t^2}{2} \hat{j} + \frac{t^3}{3} \hat{k} \) \quad \( t=1 \)

\[ \vec{r}'(1) = 2 \hat{i} + \frac{1}{3} \hat{j} + \frac{1}{3} \hat{k} \]

\[ \vec{r}''(1) = \hat{i} + \frac{2}{3} \hat{j} + \hat{k} \]

\[ \vec{r}'''(1) = \hat{i} + \frac{3}{2} \hat{j} + \hat{k} \]

\[ |\vec{r}'(1)| = (1 + 1 + 1)^{\frac{1}{2}} = 2 \quad \text{speed} \]

\[ \text{direction: } \frac{\vec{r}'(1)}{|\vec{r}'(1)|} = \frac{\hat{i} + \frac{3}{2} \hat{j} + \hat{k}}{2} \]

\[ = \frac{1}{2} \hat{i} + \frac{3}{4} \hat{j} + \frac{1}{2} \hat{k} \]

\[ \vec{r}'(1) = (2) \left( \frac{1}{2} \hat{i} + \frac{1}{4} \hat{j} + \frac{1}{4} \hat{k} \right) \]
\[ \text{speed} \quad \text{direction} \]
13.2 Projectile Motion

Usually this means 2-D motion under constant acceleration (pointing down)

\[ \vec{r}(t) = \text{position of object at time } t \]
\[ \vec{r}(0) = \vec{r}_0 \quad \text{initial position} \]
\[ \vec{v}(t) = \vec{r}'(t) = \text{velocity of object at time } t \]
\[ \vec{v}(0) = \vec{v}_0 = \text{initial velocity} \]
\[ g = \text{acceleration due to gravity} \]
\[ g = 32 \text{ ft/sec}^2 \quad g = 9.8 \text{ m/s}^2 \]
\[ \vec{r}''(t) = -g \vec{j} \]
\[ \vec{r}'(t) = -gt \vec{j} + \vec{v}_0 + \vec{c} \]
\[ \vec{r}'(0) = \vec{v}_0 \quad \vec{c} = \vec{v}_0 \]
\[ \vec{r}(0) = \vec{c} \]
\[ \vec{r}(t) = -\frac{1}{2}gt^2 \vec{j} + \vec{v}_0 t + \vec{c} \]
\[ \vec{v}(0) = \vec{r}_0 \quad \vec{r}_0 = \vec{c} \]
\[ \vec{v}(t) = -\frac{1}{2}gt^2 \vec{j} + \vec{v}_0 t + \vec{v}_0 \]
With 2-d vectors we write:

\[ \vec{v}_0 = (\vec{v}_0 \cos \alpha) \hat{i} + (\vec{v}_0 \sin \alpha) \hat{j} \]

\[ \alpha = \text{angle of elevation} \]
\[ \vec{v}_0 = \text{initial speed}. \]

1. Look at examples where initial position is \( \vec{r} \):

\[ \vec{r}(t) = -\frac{1}{2}gt^2 \hat{j} + \vec{v}_0 \hat{t} \]

\[ = -\frac{1}{2}gt^2 \hat{j} + (\vec{v}_0 \cos \alpha) \hat{i} \]

\[ + (\vec{v}_0 \sin \alpha) \hat{j} \]

\[ = \vec{r}(0) + \vec{v}_0 \hat{t} \]

\[ + \left( \frac{\vec{v}_0^2 \sin \alpha}{\hat{j}} + \left( -\frac{1}{2}gt^2 + \vec{v}_0 \sin \alpha \right) \hat{j} \right) \]

*Note: In horizontal direction, constant speed. Not affected by gravity.*