No class Monday
Exam solutions will be posted
Exams returned Tuesday
MAPLE #1 is posted online

7.2 Exponential growth/decay

Idea: Exp growth/decay are good models for many situations

1. Models well very rapid growth
2. Characterized by a doubling time (for growth models) or half-life (for decay models).

\[ y(t) = \text{population or amount, etc. at time } t. \]

Then there is a fixed value \( T \) so that

\[ y(t+T) = 2y(t) \quad \text{(growth)} \]
\[ y(t+T) = \frac{1}{2} y(t) \quad \text{(decay)} \]

3. The rate of change in \( y \), \( y'(t) \)
   is proportional to \( y(t) \)
Suppose \( y'(t) \) proportional to \( y(t) \)

\[
y'(t) = ky(t)
\]

\[
\frac{y'(t)}{y(t)} = k
\]

\[
\int \frac{y'(t)}{y(t)} \, dt = \int k \, dt
\]

\[
\ln y(t) = kt + c
\]

\[
y(t) = e^{kt+c} = e^{kt}e^c
\]

\[
y(t) = y_0 e^{kt}
\]

\( y_0 = y(0) \) = initial value

\[
\text{Call this } y_0
\]

\( k \): growth/decay constant

\( k > 0 \) growth

\( k < 0 \) decay
y(t) = pop. of bacteria t hours after initial.

Want y(0) = y_0.

Know y(t) = y_0 e^kt for some k > 0, y_0.

y(3) = 10000
y(5) = 40000

Two ways to solve:

1. Pop has increased by factor of 4 between t = 3 and t = 5. In fact, any 2 hr time period increases pop. by factor of 4.

Pop doubles twice in 2 hrs so will double once in one hour.

y(4) = 2y(3) = 20000
y(2) = 5000
y(1) = 2500
y(0) = 1250

-3-
2. \( y(t) = y_0 e^{kt} \) \text{ Find } k + y_0 \\

\( y(3) = 10000 \) \\
\( y(5) = 40000 \) \\
\( y_0 e^{k \cdot 3} = 10000 \) \\
\( y_0 e^{k \cdot 5} = 40000 \) \\
\( 4 y_0 e^{3k} = y_0 e^{5k} \) \\
\( 4 = e^{5k} e^{-3k} = e^{2k} \) \\
\( \ln(4) = 2k \) \quad \therefore \quad k = \frac{\ln(4)}{2} = \frac{1}{2} \ln(4) \) \\
\( = \ln(4^{1/2}) = \ln(2) \) \\

\( y(t) = y_0 e^{\ln(2)t} \) \\

\text{Find } y_0: \quad 10000 = y_0 e^{3 \ln(2)} = y_0 e^{\ln(8)} = y_0 e \) \\
\( y_0 = \frac{10000}{8} = 1250 \)
Example 1, p510

\[ y(t) = y_0 e^{kt} \quad k < 0 \]

\[ y(0) = y_0 \]

\[ y(1) = 0.8 y_0 \]

\[ y(2) = (0.8)^2 y_0 = 0.64 y_0 \]

\[ \text{etc...} \]

\[ y_0 = 10000 \quad (\text{given}) \]

Find \( k \).

\[ y(1) = y_0 e^{k} \quad y(1) = 0.8 y_0 \]

\[ 0.8 y_0 = y_0 e^{k} \]

\[ 0.8 = e^{k} \quad k = \ln(0.8) \]

\[ y(t) = 10000 e^{\ln(0.8) t} \]

\[ \approx -0.223 \]

Find \( t \) so that \( y(t) = 1000 \)

\[ 1000 = 10000 e^{\ln(0.8) t} \]

\[ e^{\ln(0.8) t} = 0.1 \]

\[ \ln(0.8) t = \ln(0.1) \]

\[ t = \frac{\ln(0.1)}{\ln(0.8)} \approx 10.3 \text{ yrs.} \]
(a) \[ y(t) = y_0 e^{kt} \]
\[ y_0 = 10000 \text{ (still)} \]
\[ \text{be different} \]

\[ y(1) = .75y_0 \]
\[ y(1) = y_0 e^k \]
\[ .75y_0 = y_0 e^k \]
\[ e^k = .75 \]
\[ k_0 = \ln(.75) \approx -.287 \]

When will \( y(t) = 1000 \) ?

\[ y(t) = 10000 e^{\ln(.75)t} \]
\[ e^{\ln(.75)t} = .1 \]
\[ \ln(.75)t = \ln(.1) \]
\[ t = \frac{\ln(.1)}{\ln(.75)} \approx 8.0 \text{ yrs.} \]
\#6 515  \[ \frac{dV}{dt} = -\frac{1}{40} \ V \]

\[ V(t) = V_0 \ e^{(-\frac{1}{40})t} \]

Solve: \[ V(t) = 0.1 \ V_0 \]

\[ e^{(-\frac{1}{40})t} = 0.1 \]

\[ -\frac{1}{40}t = \ln(0.1) \]

\[ t = -40 \ \ln(0.1) \approx 92.1 \ \text{sec} \]
2.4 Hyperbolic Functions

Idea: Will allow us to do more integrals.

Hyperbolic sine: \( \sinh(x) = \frac{e^x - e^{-x}}{2} \)

Hyperbolic cosine: \( \cosh(x) = \frac{e^x + e^{-x}}{2} \)

Hyperbolic tangent: \( \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \)

etc...

Basic properties:

\( \sinh(x) : \sinh(-x) = \frac{e^{-x} - e^x}{2} = -(\frac{e^x - e^{-x}}{2}) = -\sinh(x) \quad \text{ODD} \)

\( \cosh(x) : \cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh(x) \quad \text{EVEN} \)

\( \sinh(x) + \cosh(x) = e^x \)

\( \text{odd part of } e^x \quad \text{even part of } e^x \)
Graphs:
\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]
\[ \sinh(0) = 0 \]

\[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]
\[ \cosh(0) = 1 \]

\[ \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]
\[ \tanh(0) = 0 \]
Identities:
\[ \sinh(x) + \cosh(x) = e^x \]
\[ \cosh^2(x) - \sinh^2(x) = 1 \]
\[ \tanh^2(x) = 1 - \text{sech}^2(x) \]
\[ \coth^2(x) = 1 + \text{csch}^2(x) \]

Derivatives:
\[ \frac{d}{dx} \sinh(x) = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x) \]
\[ \frac{d}{dx} \cosh(x) = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh(x) \]

etc...
Inverse functions

\[ y = f(x) \]

If \( f \) is one-to-one
the function \( f^{-1} \) exists and is defined by:
\[ f^{-1}(x) = y \iff f(y) = x \]

Fact: if \( (x, y) \) is on graph of \( f \) then \( (y, x) \) is on graph of \( f^{-1}(x) \)
\[
\sinh^{-1}(x)\\
\sinh^{-1}(y) = y \iff \sinh(y) = x
\]

In fact you can solve \( \sinh(x) = y \) for \( x \) in terms of \( y \).

\[
y = \frac{e^x - e^{-x}}{2} \quad \longrightarrow \quad x = \ln(y + \sqrt{y^2 + 1}) = \sinh^{-1}(y).
\]

\[
\cosh^{-1}(x)\\
\cosh^{-1}(x) = y \iff x = 1 + \sqrt{x^2 - 1}, \quad x \geq 1
\]

\[
\text{NOT one-to-one}
\]
\[
\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\cosh(\sinh^{-1}(x))} \\
= \frac{1}{\sqrt{1+x^2}}
\]

Similarly,

\[
\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}
\]

\[
\frac{d}{dx} \tanh^{-1}(x) = \frac{1}{1-x^2}
\]