1.6. One-sided Limits and Continuity

One-sided Limit
If \( f(x) \) approaches \( L \) as \( x \) tends toward \( c \) from the left \( (x < c) \), we write
\[
\lim_{{x \to c^-}} f(x) = L.
\]
Likewise, if \( f(x) \) approaches \( M \) as \( x \) tends toward \( c \) from the right \( (x > c) \), then
\[
\lim_{{x \to c^+}} f(x) = M.
\]

Example
Find \( \lim_{{x \to 2^-}} f(x) \) and \( \lim_{{x \to 2^+}} f(x) \) for the function
\[
f(x) = \frac{x^2 + 3}{x - 2}
\]

\[
\lim_{{x \to 2^-}} \frac{x^2 + 3}{x - 2} = -\infty
\]
Vertical asymptote
\( x < 2 \)
\( x^2 + 3 > 0 \) (1)
\( x - 2 < 0 \) (2) = negative.
\[
\lim_{x \to c^-} f(x) = L
\]

\[
\lim_{x \to c^+} f(x) = L
\]

\[
\lim_{x \to c} f(x) \ does \ not \ exist.
\]
\[\lim_{x \to 2^+} \frac{x^2 + 3}{x - 2} = +\infty\]

**vertical asymptote**

\[x > 2 \quad x^2 + 3 > 0 \quad \frac{(+) \quad (+)}{(+) \quad (+) = \text{positive}}\]

\[\lim_{x \to +\infty} \frac{x^2 + 3}{x - 2} = \lim_{x \to +\infty} \frac{x^2}{x} = \lim_{x \to +\infty} x = +\infty\]

\[\lim_{x \to -\infty} \frac{x^2 + 3}{x - 2} = \lim_{x \to -\infty} x = -\infty\]
One-sided Limit

Example

Find \( \lim_{x \to -1^-} f(x) \) and \( \lim_{x \to -1^+} f(x) \) for the function

\[
f(x) = \begin{cases} 
\frac{2}{x-1} & \text{if } x < -1 \\
x^2 - x & \text{if } x \geq -1 
\end{cases}
\]

\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{2}{x-1} = \frac{2}{-2} = -1
\]

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 - x) = (-1)^2 - (-1) = 2
\]

\[
\lim_{x \to -1} f(x) \text{ DOES NOT EXIST}
\]
Existence of a Limit

Theorem

The two-sided limit \( \lim_{x \to c} f(x) \) exists if and only if the two one-sided limits \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) both exist and are equal, and then

\[
\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)
\]

Example

Determine whether \( \lim_{x \to 1} f(x) \) exists, where

\[
f(x) = \begin{cases} 
2x + 1 & \text{if } x < 1 \\
-x^2 + 2x + 2 & \text{if } x \geq 1
\end{cases}
\]

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-x^2 + 2x + 2) = 3
\]

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 1) = 3
\]

\[
\therefore \lim_{x \to 1} f(x) = 3
\]
Continuity

A function \( f \) is continuous at \( c \) if all three of these conditions are satisfied:

a. \( f(c) \) is defined
b. \( \lim_{{x \to c}} f(x) \) exists
   \[ \lim_{{x \to c}} f(x) = L \]
c. \( \lim_{{x \to c}} f(x) = f(c) \)

If \( f(x) \) is not continuous at \( c \), it is said to have a discontinuity there.

Example

Decide if \( f(x) = x^3 - x^2 + x - 4 \) is continuous at \( x = 0 \).
Continuity

Example
Decide if \( f(x) = \frac{2x + 5}{2x - 4} \) is continuous at \( x = 2 \).

\[ f(2) \text{ not defined so } NO \]
Continuity

Continuity of Polynomials and Rational Functions
A polynomial or a rational function is continuous wherever it is defined.

Example
List all values of $x$ for which $f(x)$ is not continuous

$$f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}$$
Continuity

Example

Decide if \( f(x) = \begin{cases} 
    x + 1 & \text{if } x < 0 \\
    x - 1 & \text{if } x \geq 0
\end{cases} \) is continuous at \( x = 0 \).

\( f(0) = -1 \)  \( \text{N} \)

\( \lim_{x \to 0} f(x) \) DNE

\( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 1 = -1 \)

\( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x + 1 = 1 \)
Continuity

Example
Find the value of the constant $A$ such that the function

$$f(x) = \begin{cases} 
1 - 2x & \text{if } x < 2 \\
Ax^2 + 2x - 3 & \text{if } x \geq 2 
\end{cases}$$

will be continuous for all $x$. 
Intermediate Value Property

The intermediate value property

If \( f(x) \) is continuous on the interval \( a \leq x \leq b \) and \( L \) is a number between \( f(a) \) and \( f(b) \), the \( f(c) = L \) for some number \( c \) between \( a \) and \( b \). In other words, a continuous function attains all values between any two of its values.

Example

Show that the equation \( \sqrt[3]{x} = x^2 + 2x - 1 \) must have at least one solution on the interval \( 0 \leq x \leq 1 \).