Approximation by Increments

If $f(x)$ is differentiable at $x = x_0$ and $\Delta x$ is a small change in $x$, then

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

or, equivalently, if $\Delta f = f(x_0 + \Delta x) - f(x_0)$, then

$$\Delta f \approx f'(x)\Delta x$$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

if $\Delta x$ small

$$\Delta x f'(x_0) \approx f(x_0 + \Delta x) - f(x_0) = \Delta f$$

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$
\[ f(x) = \frac{x}{x+1} - 3 \]

Estimate \( \Delta f \) as \( x \) changes from 4 to 3.8

\[
\begin{align*}
\frac{x+\Delta x}{x+1} & \approx \frac{x}{x+1} \\
\therefore \Delta x & = -0.2
\end{align*}
\]

\[
\Delta f = f(3.8) - f(4) = f(x+\Delta x) - f(x)
\]

\[
\Delta f \approx f'(x) \Delta x = f'(4) (-0.2)
\]

\[
f'(x) = \frac{(x+1)(1)-(x)(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}
\]

\[
f'(4) = \frac{1}{(4+1)^2} = \frac{1}{25}
\]

\[
\Delta f \approx \left(\frac{1}{25}\right) (-0.2) = (0.04) (-0.2) = -0.008
\]

\[
\Delta f = f(3.8) - f(4) = \left(\frac{3.8}{4.8} - 3\right) - \left(\frac{4}{5} - 3\right)
\]

\[
f(3.8) = \frac{3.8}{3.8+1} - 3 \approx \frac{3.8}{4.8} - 3
\]

\[
\approx 0.791667 - 0.8 = -0.008333\ldots
\]
Approximation by Increments

Example
A 5-year projection of population trends suggests that $t$ years from now, the population of certain community will be

$$P(t) = -t^3 + 9t^2 + 48t + 200 \text{ thousand.}$$

(a) Find the rate of change of population $R(t) = P'(t)$ with respect to time $t$.

(b) At what rate does the population growth rate $R(t)$ change with respect to time? $P''(t)$

(c) Use increments to estimate how much $R(t)$ changes during the first month of the fourth year. What is the actual change in $R(t)$ during this time period?

\[ R(t) = P'(t) = -3t^2 + 18t + 48 \text{ thousands per year.} \]

\[ P'(4) = -3(4)^2 + 18(4) + 48 = -48 + 72 + 48 = 72 \text{ thousands per year.} \]

\[ \Delta P \approx P'(4) \left( \frac{1}{12} \right) = (72) \left( \frac{1}{12} \right) = 6 \text{ thousand.} \]
Exact change: $P(4 + \frac{1}{2}) - P(4)$

$= P(4.0833\ldots) - P(4)$

$\approx 478 - 472 = 6$  

Estimate was very good.
Approximation by Increments

Approximation formula for Percentage Change
If $\Delta x$ is a (small) change in $x$, the corresponding percentage change in the function $f(x)$ is

$$\text{Percentage change in } f = 100 \frac{\Delta f}{f(x)} \approx 100 \frac{f'(x) \Delta x}{f(x)}$$

Example
Use increments to estimate the percentage change in the function $f(x) = 3x + \frac{2}{x}$ as $x$ decreases from 5 to 4.6.
1500 = x^2 y

\[ S = x^2 + xy + xy + xy + xy + xy + x^2 = 2x^2 + 4xy \]

\[ y = \frac{1500}{x^2} \]

\[ S = 2x^2 + 4x \left( \frac{1500}{x^2} \right) = 2x^2 + \frac{6000}{x} \]
#4)  

X - one number  

Y - the other  

\[ x + y = 18 \rightarrow y = 18 - x \]  

P - product of #s  

\[ p = x \cdot y \]  

\[ p = x(18 - x) \checkmark \]

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\[ S = 2\pi rh \]  

\[ 2\pi r^2 + 2\pi rh = 2\pi rh + \pi r^2 \]  

\[ h = \frac{2\pi - \pi r^2}{2\pi} \]  

\[ V = \pi r^2 h = \pi r^2 \left( \frac{2\pi - r^2}{2\pi} \right) = \pi r(2\pi - r^2) \]