2.1 The Derivative

The derivative of a function

The *derivative* of the function $f(x)$ with respect to $x$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$ 

The process of computing the derivative is called *differentiation*, and we say that $f(x)$ is *differentiable* at $x = c$ if $f'(c)$ exists.

**Example**

Find the derivative of the function $f(x) = x^2 - 2x$. 
Slope as a Derivative

The slope of the tangent line to the curve \( y = f(x) \) at the point \((c, f(c))\) is \( m_{\text{tan}} = f'(c) \).

Example
Find the equation of the tangent line to the curve \( y = x^2 - 2x \) at the point where \( x = -1 \).
Instantaneous Rate of Change as a Derivative

The rate of change of \( f(x) \) with respect to \( x \) when \( x = c \) is given by \( f'(c) \).

Example

A toy rocket rises vertically in such a way that \( t \) seconds after lift-off, it is

\[
h(t) = -\frac{1}{2} t^2 + 20t
\]

feet above ground.

a. What is the (instantaneous) velocity of the rocket at lift-off?

b. What is its velocity after 10 seconds?
Significance of the sign of $f'(x)$

If the function $f$ is differentiable at $x = c$, then

$$f \text{ is } increasing \text{ at } x = c \text{ if } f'(c) > 0$$

and

$$f \text{ is } decreasing \text{ at } x = c \text{ if } f'(c) < 0$$

Example

(c. At lift-off, is the rocket rising?)

(d. Is the rocket rising after 30 seconds?)
Derivative Notation

The derivative \( f'(x) \) of \( y = f(x) \) is sometimes written as

\[
\frac{dy}{dx} \quad \text{or} \quad \frac{df}{dx}
\]

In this notation, \( f'(c) \) is written as

\[
\frac{dy}{dx} \bigg|_{x=c} \quad \text{or} \quad \frac{df}{dx} \bigg|_{x=c}
\]

Example

Find the rate of change \( \frac{dy}{dx} \) of \( y = 5 - x^2 \) at the point where \( x = 2 \).
Differentiability and Continuity

Continuity of a differentiable function
If the function $f(x)$ is differentiable at $x = c$, then it is also continuous at $x = c$. This means that for $f(x)$ to be differentiable at $x = c$ it must at least be continuous there, but more is required. There are functions that are continuous at a point but not differentiable there.

Examples of nondifferentiability
Each of the functions below is continuous at $x = 0$ but not differentiable at $x = 0$.

- Vertical tangent: $f(x) = x^{1/3}$
- Cusp: $f(x) = x^{2/3}$
- Corner: $f(x) = |x|$