1.6. One-sided Limits and Continuity

One-sided Limit
If \( f(x) \) approaches \( L \) as \( x \) tends toward \( c \) from the left \( (x < c) \), we write
\[
\lim_{x \to c^-} f(x) = L.
\]
Likewise, if \( f(x) \) approaches \( M \) as \( x \) tends toward \( c \) from the right \( (x > c) \), then
\[
\lim_{x \to c^+} f(x) = M.
\]

Example
Find \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \) for the function
\[
f(x) = \frac{x^2 + 3}{x - 2}
\]
Example
Find $\lim_{x\to-1^-} f(x)$ and $\lim_{x\to-1^+} f(x)$ for the function

$$f(x) = \begin{cases} 
\frac{2}{x - 1} & \text{if } x < -1 \\
2x^2 - x & \text{if } x \geq -1 
\end{cases}$$
Existence of a Limit

Theorem
The two-sided limit \( \lim_{x \to c} f(x) \) exists if and only if the two one-sided limits \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) both exist and are equal, and then

\[
\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)
\]

Example
Determine whether \( \lim_{x \to 1} f(x) \) exists, where

\[
f(x) = \begin{cases} 
  2x + 1 & \text{if } x < 1 \\
  -x^2 + 2x + 2 & \text{if } x \geq 1
\end{cases}
\]
Continuity

A function $f$ is *continuous* at $c$ if all three of these conditions are satisfied:

a. $f(c)$ is defined

b. $\lim_{x \to c} f(x)$ exists

c. $\lim_{x \to c} f(x) = f(c)$

If $f(x)$ is not continuous at $c$, it is said to have a *discontinuity* there.

**Example**

Decide if $f(x) = x^3 - x^2 + x - 4$ is continuous at $x = 0$. 
Continuity

Example

Decide if \( f(x) = \frac{2x + 5}{2x - 4} \) is continuous at \( x = 2 \).
Continuity

Continuity of Polynomials and Rational Functions
A polynomial or a rational function is continuous \textit{wherever it is defined}.

Example
List all values of $x$ for which $f(x)$ is not continuous

\[
f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}
\]
Continuity

Example

Decide if \( f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases} \) is continuous at \( x = 0 \).
Example
Find the value of the constant $A$ such that the function

$$f(x) = \begin{cases} 
1 - 2x & \text{if } x < 2 \\
A x^2 + 2x - 3 & \text{if } x \geq 2
\end{cases}$$

will be continuous for all $x$. 
Intermediate Value Property

The intermediate value property

If \( f(x) \) is continuous on the interval \( a \leq x \leq b \) and \( L \) is a number between \( f(a) \) and \( f(b) \), the \( f(c) = L \) for some number \( c \) between \( a \) and \( b \). In other words, a continuous function attains all values between any two of its values.

Example

Show that the equation \( \sqrt[3]{x} = x^2 + 2x - 1 \) must have at least one solution on the interval \( 0 \leq x \leq 1 \).