1.1. Functions

Definition

- A function is a rule that assigns to each object in a set $A$ exactly one object in a set $B$.
- The set $A$ is called the domain of the function.
- The set of assigned objects in $B$ is called the range of the function.

For the purposes of this class, the following will always be true.

- $A$ and $B$ will always be subsets of the real numbers $\mathbb{R}$.
- A function will be denoted by $f(x)$, and $f(x)$ will be given by a formula such as $f(x) = x^2 + 3$. Sometimes we write $y = f(x)$ where $x$ is the independent variable and $y$ is the dependent variable.
- The domain of $f(x)$ will be given explicitly (rarely) or will be the largest set of real numbers for which the formula for $f(x)$ makes sense (usually).
Example
Find $f(2)$ if $f(x) = x^2 + 3$.

Example
If $g(u) = (u + 1)^{3/2}$, find $g(0)$, $g(-1)$, and $g(8)$. 
Example
Find $h(2)$, $h(1)$, $h(-2)$ if

$$h(x) = \begin{cases} 
-2x + 4 & \text{if } x \leq 1 \\
x^2 + 1 & \text{if } x > 1 
\end{cases}$$
Examples: Finding the Domain

a. \( f(t) = \frac{t + 3}{t^2 - t - 2} \)

b. \( h(x) = \sqrt{x^2 - 4} \)
Composition of functions

Definition
Given functions \( f(u) \) and \( g(x) \), the composition \( f(g(x)) \) is the function of \( x \) formed by substituting \( u = g(x) \) for \( u \) in the formula for \( f(u) \).

Example
Find the composite function \( f(g(x)) \), where \( f(u) = u^2 + 3 \) and \( g(x) = x - 1 \).
Example

Find the composite functions $f(g(x))$ and $g(f(x))$, where $f(x) = x^2 + 3x + 1$ and $g(x) = 1 + x$. Note that $f(g(x)) \neq g(f(x))$. 
Example
At a certain factory, the total cost of manufacturing $q$ units during the daily production run is $C(q) = q^2 + q + 900$ dollars. On a typical workday, $q(t) = 25t$ units are manufactured during the first $t$ hours of a production run.

a. Express the total manufacturing cost as a function of $t$.

b. How much will have been spent on production by the end of the third hour?

c. When will the total manufacturing cost reach $11,000$?
Difference quotient

Definition
A *difference quotient* is an expression of the general form

\[
\frac{f(x + h) - f(x)}{h}
\]

where \( f \) is a function of \( x \) and \( h \) is a number.

Example
Find the difference quotient for \( f(x) = 2x - x^2 \).