Exam 1 - Feb 21

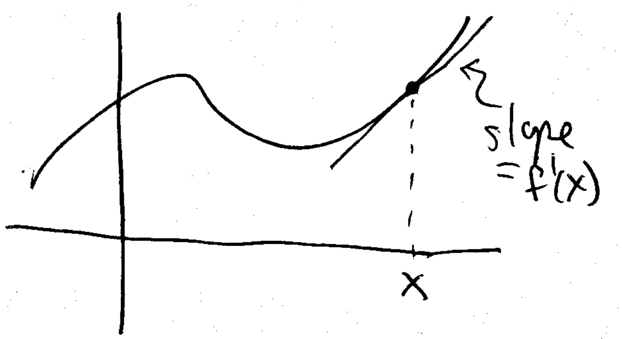
Coverage will be Sections 1.4-1.6, Chap 2.

$f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

① Derivative is the limit of the difference quotient

② Instantaneous rate of change of f with respect to x.



③ slope of tangent line to graph of $f(x)$ at x.

2.3. Product and Quotient Rules; Higher-Order Derivatives

The Product Rule

If $f(x)$ and $g(x)$ are differentiable at x, then so is their product and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

or equivalently

$$(fg)' = fg' + gf'$$

Know:

$$\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g$$

Not ~~true~~

$$\cancel{\frac{d}{dx}(fg) = \frac{d}{dx}f \cdot \frac{d}{dx}g}$$

Example

Differentiate $f(x) = (2x - 5)(1 - x)$.

$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left((2x-5)(1-x)\right) \\
&= (2x-5)\frac{d}{dx}(1-x) + (1-x)\frac{d}{dx}(2x-5) \\
&= (2x-5)(-1) + (1-x)(2) \\
&= -2x+5+2-2x \\
&= -4x+7
\end{aligned}$$

Another way:

$$f(x) = (2x-5)(1-x) = -2x^2+7x-5$$

$$f'(x) = (-2)(2x)+7(1) = -4x+7$$

The Product Rule

Example

Differentiate $f(x) = (x^3 - 2x^2 + 5)(\sqrt{x} + 2x)$.

$$f'(x) = (x^3 - 2x^2 + 5)\frac{d}{dx}(x^{1/2} + 2x) + (x^{1/2} + 2x)\frac{d}{dx}(x^3 - 2x^2 + 5)$$

$$= (x^3 - 2x^2 + 5)\left(\frac{1}{2}x^{-1/2} + 2\right) + (x^{1/2} + 2x)(3x^2 - 4x)$$

$$= \frac{1}{2}x^{5/2} + 2x^3 - x^{3/2} - 4x^2 + \frac{5}{2}x^{-1/2} + 10 + 3x^{5/2} - 4x^{3/2} + 6x^3 - 8x^2$$

$$= \frac{7}{2}x^{5/2} + 8x^3 - 5x^{3/2} - 12x^2 + \frac{5}{2}x^{-1/2} + 10$$

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions, then so is the quotient $Q(x) = f(x)/g(x)$ and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g^2(x)}$$

or equivalently

$$\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

Example

Differentiate $y = \dfrac{1+x^2}{1-x^2}$.

$$\frac{dy}{dx} = \frac{(1-x^2)\frac{d}{dx}(1+x^2) - (1+x^2)\frac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$= \frac{(1-x^2)(2x) - (1+x^2)(-2x)}{(1-x^2)^2}$$

$$= \frac{(2x-2x^3)-(-2x-2x^3)}{(1-x^2)^2}$$

$$= \frac{2x - \cancel{2x^3} + 2x + \cancel{2x^3}}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2} \mathbin{//}$$

The Quotient Rule

Example

Find all points on the graph of $f(x) = \dfrac{x^2 + x - 1}{x^2 - x + 1}$ where the tangent line is horizontal.

Strategy: tangent line horizontal means slope $= 0$.
Look for x where $f'(x) = 0$.
Find $f'(x)$. Then solve $f'(x) = 0$.

$$f'(x) = \frac{(x^2-x+1)\frac{d}{dx}(x^2+x-1) - (x^2+x-1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$$

$$= \frac{(x^2-x+1)(2x+1) - (x^2+x-1)(2x-1)}{(x^2-x+1)^2}$$

$$= \frac{(2x^3-x^2+x+1) - (2x^3+x^2-3x+1)}{(x^2-x+1)^2}$$

$$= \frac{\cancel{2x^3}-x^2+x+\cancel{1}-\cancel{2x^3}-x^2+3x-\cancel{1}}{(x^2-x+1)^2}$$

$$= \frac{-2x^2+4x}{(x^2-x+1)^2}$$

Set $f'(x) = 0$ + solve

$$\frac{-2x^2+4x}{(x^2-x+1)^2} = 0$$

$$-2x^2+4x = 0$$

$$-2x(x-2) = 0$$

$x = 0$ $x = 2$

$f(0) = -1$

$$f(2) = \frac{4+2-1}{4-2+1} = \frac{5}{3}$$

Points: $(0,-1)$

$(2, 5/3)$

Product rule and Quotient Rule

Example

Differentiate $g(x) = \dfrac{(x^2 + x + 1)(4 - x)}{2x - 1}$.

$$g'(x) = \frac{(2x-1)\frac{d}{dx}[(x^2+x+1)(4-x)] - (x^2+x+1)(4-x)(2)}{(2x-1)^2}$$

$$= \frac{(2x-1)[(x^2+x+1)(-1)+(4-x)(2x+1)] - (x^2+x+1)(4-x)(2)}{(2x-1)^2}$$

$$= \frac{(2x-1)[-x^2-x-1-2x^2+7x+4] - (x^2+x+1)(8-2x)}{(2x-1)^2}$$

$$= \frac{(2x-1)(-3x^2+6x+3) - (-2x^3+6x^2+6x+8)}{(2x-1)^2}$$

$$= \frac{-6x^3+15x^2-3+2x^3-6x^2-6x-8}{(2x-1)^2}$$

$$= \frac{-4x^3+9x^2-6x-11}{(2x-1)^2}$$

The Second Derivative

The second derivative of a function is the derivative of its derivative. If $y = f(x)$, the second derivative is denoted by

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x) \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

The second derivative gives the rate of change of the rate of change of the original function.

Example

Find the second derivative of $f(x) = x^{10} - 4x^6 - 27x + 4$.

$$f'(x) = 10x^9 - 24x^5 - 27$$

$$f''(x) = 90x^8 - 120x^4 \,//$$

The Second Derivative

Example

Find the second derivative of $y = (x^2 - 2x)\left(x - \frac{1}{x}\right)$.

2.4. The Chain Rule

$f(g(x)) = f \circ g(x)$

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is in turn a differentiable function of x, then the composite function $\boxed{f(g(x))}$ is a differentiable function of x whose derivative is given by the product

$$\boxed{\frac{dy}{dx} = \frac{dy}{\cancel{du}}\frac{\cancel{du}}{dx}}$$

$u = g(x)$

$\underbrace{y = f(u)}$

$y = f(g(x))$

or, equivalently, by

$$\frac{dy}{dx} = \underbrace{f'(g(x))}_{\frac{dy}{du}}\underbrace{g'(x)}_{\frac{dy}{dx}},$$

The Chain Rule

Example

Compute the derivative $\frac{dy}{dx}$ and simplify the answer if

$$y = u^2 - 3u + 4; \quad u = 1 - x^2$$

$$y = (1-x^2)^2 - 3(1-x^2) + 4$$
$$= (1 - 2x^2 + x^4) - 3 + 3x^2 + 4$$
$$= x^4 + x^2 + 2$$
$$\frac{dy}{dx} = 4x^3 + 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (2u-3)(-2x)$$
$$= (2(1-x^2)-3)(-2x)$$
$$= (2-2x^2-3)(-2x)$$
$$= (-2x^2-1)(-2x)$$
$$= 4x^3 + 2x$$

The Chain Rule

Example

Compute the derivative $\left.\dfrac{dy}{dx}\right|_{x=\frac{1}{2}}$ if

$$y = u^2 - 2u + 2; \quad u = \frac{1}{x}$$

The Chain Rule

Sometimes when dealing with a composite function $y = f(g(x))$ it may help to think of f as the "outer" function and g as the "inner" function. Then the chain rule says that the derivative of $y = f(g(x))$ with respect to x is given by *the derivative of the outer function evaluated at the inner function times the derivative of the inner function.* $\frac{dy}{dx} = f'(g(x))\,g'(x)$

Example

Differentiate the following function and simplify the answer.

$$h(x) = \sqrt{x^6 - 3x^2}$$

$f(u) = u^{1/2}$

$g(x) = x^6 - 3x^2$

$f'(u) = \frac{1}{2}u^{-1/2}$

$6x^5 - 6x = g'(x)$

$$h(x) = (x^6 - 3x^2)^{1/2}$$

$$= \frac{1}{2}(x^6 - 3x^2)^{-1/2}(6x^5 - 6x)$$

The General Power Rule

For any real number n and differentiable function h,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1} \cdot \frac{d}{dx}[h(x)]$$

Example

Differentiate the following function and simplify the answer.

$$f(\cancel{x}t) = (t^4 - 4t^2 + 4)^6$$

$$f'(t) = 6(t^4 - 4t^2 + 4)^5 (4t^3 - 8t)$$