Exam I - Feb 21

Coverage will be Sections 1.4 - 1.6, Chap 2.

\[ f(x) \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \]

1. Derivative is the limit of the difference quotient.
2. Instantaneous rate of change of \( f \) with respect to \( x \).
3. Slope of tangent line to graph of \( f(x) \) at \( x \).
2.3. Product and Quotient Rules; Higher-Order Derivatives

The Product Rule
If \( f(x) \) and \( g(x) \) are differentiable at \( x \), then so is their product and
\[
\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\]
or equivalently
\[
(fg)' = fg' + gf'
\]

Example
Differentiate \( f(x) = (2x - 5)(1 - x) \).

\[
\frac{df}{dx} = \frac{d}{dx} \left( (2x - 5)(1 - x) \right)
\]
\[
= (2x - 5)\frac{d}{dx}(1 - x) + (1 - x)\frac{d}{dx}(2x - 5)
\]
\[
= (2x - 5)(-1) + (1 - x)(2)
\]
\[
= -2x + 5 + 2 - 2x
\]
\[
= -4x + 7
\]
Another way:

\[ f(x) = (2x-5)(1-x) = -2x^2 + 7x - 5 \]

\[ f'(x) = (-2)(2x) + 7(1) = -4x + 7 \]
The Product Rule

Example
Differentiate \( f(x) = (x^3 - 2x^2 + 5)(\sqrt{x} + 2x) \).

\[
\begin{align*}
 f'(x) &= (x^3 - 2x^2 + 5) \frac{d}{dx} (\sqrt{x} + 2x) + (\sqrt{x} + 2x) \frac{d}{dx} (x^3 - 2x^2 + 5) \\
 &= (x^3 - 2x^2 + 5) \left( \frac{1}{2} x^{-1/2} + 2 \right) + (\sqrt{x} + 2x) (3x^2 - 4x) \\
 &= \frac{1}{2} x^{5/2} + 2x^3 - x^{3/2} - 4x^2 + \frac{5}{2} x^{-1/2} + 10 + 3\sqrt{x} - 4x \sqrt{x} + 6x^2 - 8x \\
 &= \frac{7}{2} x^{5/2} + 8x^3 - 5x^{3/2} - 12x^2 + \frac{5}{2} x^{-1/2} + 10
\end{align*}
\]
The Quotient Rule

If \( f(x) \) and \( g(x) \) are differentiable functions, then so is the quotient \( Q(x) = \frac{f(x)}{g(x)} \) and

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}[f(x)] - f(x) \frac{d}{dx}[g(x)]}{g^2(x)}
\]

or equivalently

\[
\left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}
\]

Example

Differentiate \( y = \frac{1 + x^2}{1 - x^2} \).

\[
\frac{dy}{dx} = \frac{(1-x^2) \frac{d}{dx}(1+x^2) - (1+x^2) \frac{d}{dx}(1-x^2)}{(1-x^2)^2} = \frac{(1-x^2)(2x) - (1+x^2)(-2x)}{(1-x^2)^2} = \frac{2x - 2x^3 - (-2x - 2x^3)}{(1-x^2)^2} = \frac{2x - 2x^3 + 2x + 2x^3}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}
\]
The Quotient Rule

Example

Find all points on the graph of \( f(x) = \frac{x^2 + x - 1}{x^2 - x + 1} \) where the tangent line is horizontal.

**Strategy:** Tangent line horizontal means slope = 0, look for \( x \) where \( f'(x) = 0 \).

Find \( f'(x) \). Then solve \( f'(x) = 0 \).

\[
f'(x) = \frac{(x^2-x+1) \frac{d}{dx}(x^2+x-1) - (x^2+x-1) \frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}
\]

\[
= \frac{(x^2-x+1)(2x+1) - (x^2+x-1)(2x-1)}{(x^2-x+1)^2}
\]

\[
= \frac{(2x^3-x^2+x+1) - (2x^3+x^2-3x+1)}{(x^2-x+1)^2}
\]

\[
= \frac{2x^3-x^2+x+1-2x^3-x^2+3x-1}{(x^2-x+1)^2}
\]

\[
= \frac{-2x^2+4x}{(x^2-x+1)^2}
\]
Set \( f'(x) = 0 \) and solve

\[
\frac{-2x^2 + 4x}{(x^2 - x + 1)^2} = 0
\]

\[-2x^2 + 4x = 0
\]

\[-2x(x - 2) = 0
\]

\[x = 0 \quad x = 2 \]

\[f(0) = -1 \]

\[f(2) = \frac{4 + 2 - 1}{4 - 2 + 1} = \frac{5}{3} \]

Points: \((0, -1)\)

\((2, \frac{5}{3})\)
Product rule and Quotient Rule

Example

Differentiate $g(x) = \frac{(x^2 + x + 1)(4 - x)}{2x - 1}$.

\[
\begin{align*}
\frac{d}{dx} g(x) &= \frac{(2x-1)\frac{d}{dx}[(x^2+x+1)(4-x)] - (x^2+x+1)(4-x)(2)}{(2x-1)^2} \\
&= \frac{(2x-1)[(x^2+x+1)(-1) + (4-x)(2x+1)] - (x^2+x+1)(4-x)(2)}{(2x-1)^2} \\
&= \frac{(2x-1)[-(x^2-x-1) - 2x^2 + 7x + 4] - (x^2+x+1)(8-2x)}{(2x-1)^2} \\
&= \frac{(2x-1) (-3x^2 + 6x + 3) - (-2x^3 + 6x^2 + 6x + 8)}{(2x-1)^2} \\
&= \frac{-6x^3 + 15x^2 - 3 + 2x^3 - 6x^2 - 6x - 8}{(2x-1)^2} \\
&= \frac{-4x^3 + 9x^2 - 6x - 11}{(2x-1)^2}
\end{align*}
\]
The Second Derivative

The second derivative of a function is the derivative of its derivative. If \( y = f(x) \), the second derivative is denoted by

\[
\frac{d^2 y}{dx^2} \quad \text{or} \quad f''(x) = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}
\]

The second derivative gives the rate of change of the rate of change of the original function.

Example

Find the second derivative of \( f(x) = x^{10} - 4x^6 - 27x + 4 \).

\[
f'(x) = 10x^9 - 24x^5 - 27
\]

\[
f''(x) = 90x^8 - 120x^4
\]
The Second Derivative

Example

Find the second derivative of \( y = (x^2 - 2x) \left( x - \frac{1}{x} \right) \).
2.4. The Chain Rule

If \( y = f(u) \) is a differentiable function of \( u \) and \( u = g(x) \) is in turn a differentiable function of \( x \), then the composite function \( f(g(x)) \) is a differentiable function of \( x \) whose derivative is given by the product

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

or, equivalently, by

\[
\frac{dy}{dx} = f'(g(x))g'(x)
\]

\[
\frac{du}{dx}
\]

\[
\frac{du}{dx}
\]
The Chain Rule

Example

Compute the derivative \( \frac{dy}{dx} \) and simplify the answer if

\[
y = u^2 - 3u + 4; \quad u = 1 - x^2
\]

\[
y = (1-x^2)^2 - 3(1-x^2) + 4
\]

\[
= (1-2x^2+x^4) - 3 + 3x^2 + 4
\]

\[
= x^4 + x^2 + 2
\]

\[
\frac{dy}{dx} = 4x^3 + 2x
\]
The Chain Rule

Example
Compute the derivative \( \frac{dy}{dx} \Big|_{x=\frac{1}{2}} \) if

\[ y = u^2 - 2u + 2; \quad u = \frac{1}{x} \]
The Chain Rule

Sometimes when dealing with a composite function $y = f(g(x))$ it may help to think of $f$ as the “outer” function and $g$ as the “inner” function. Then the chain rule says that the derivative of $y = f(g(x))$ with respect to $x$ is given by the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

$$\frac{dy}{dx} = f'(g(x))g'(x)$$

Example

Differentiate the following function and simplify the answer.

$$h(x) = \sqrt{x^6 - 3x^2}$$

$$h(x) = (x^6 - 3x^2)^{\frac{1}{2}}$$

$$= \frac{1}{2} (x^6 - 3x^2)^{-\frac{1}{2}} (6x^5 - 6x)$$

$$f(u) = u^{\frac{1}{2}}$$

$$g(x) = x^6 - 3x^2$$

$$f'(u) = \frac{1}{2} u^{-\frac{1}{2}}$$

$$g'(x) = 6x^5 - 6x = g'(x)$$
The General Power Rule

For any real number $n$ and differentiable function $h$,

$$
\frac{d}{dx} [h(x)]^n = n[h(x)]^{n-1} \frac{d}{dx} [h(x)]
$$

Example

Differentiate the following function and simplify the answer.

$$
f(t) = (t^4 - 4t^2 + 4)^6
$$

$$
f'(t) = 6(t^4 - 4t^2 + 4)^5(4t^3 - 8t)
$$