Curve Sketching with the Second Derivative

Example
Determine where the function

\[ f(x) = \frac{x^2}{x^2 + 3} \]

is increasing and decreasing, and where its graph is concave up and concave down. Find all relative extrema and points of inflection, and sketch the graph.
Quiz 7: Sections 3.1, 3.2

\[ f(x) = \frac{x^2}{x^2 + 3} \]

**Intervals of increase / decrease**

\[ f'(x) = \frac{(x^2+3)(2x)-(x^2)(2x)}{(x^2+3)^2} \]

\[ = \frac{2x^3+6x-2x^3}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2} \]

Set \( f'(x) = 0 \). Solve. \( 6x = 0 \)

\( x = 0 \) \( \leftrightarrow \) critical point

**Critical point:** \( (0, f(0)) = (0, 0) \)

\[ f'(x) = \frac{6(-1)}{(-1)^2+3)^2} < 0 \]

\[ f'(1) = \frac{6(1)}{(1^2+3)^2} > 0 \]

\( (0, 0) \) \( \leftrightarrow \) local minimum
Concavity

\[ f'(x) = \frac{6x}{(x^2+3)^2} \]

\[ f''(x) = \frac{(x^2+3)^2(6) - (6x)(2(x^2+3)(2x))}{(x^2+3)^4} \]

\[ = \frac{(x^2+3)^3[6x^2+18 - 24x^2]}{(x^2+3)^4} \]

\[ = \frac{-18x^2+18}{(x^2+3)^3} = \frac{-18(x^2-1)}{(x^2+3)^3} \]

Possible inflection points:
Set \( f'' = 0 \). Solve, \(-18(x^2-1) = 0\)

\[ x^2-1 = 0 \]

\[ (x+1)(x-1) = 0 \]

\[ x = -1 \quad x = 1 \]

- - - - - 0 + + + + + + + + 0 - - - - -

\[ f'' \]

-2 -1 0 1 2

\[ f''(-2) = \frac{-18(3)}{(4+3)^3} < 0 \]

\[ f''(0) = \frac{(-18)(1)}{(3^3)} > 0 \]

\[ f''(2) = \frac{-18(3)}{(4+3)^3} 0 \]
Inflection points: \((-1, f(-1)), (1, f(1))\)

\((-1, \frac{1}{4}), (1, \frac{1}{4})\)

Horizontal asymptotes:

\[
\lim_{x \to \infty} \frac{x^2}{x^2 + 3} = 1
\]

\[
\lim_{x \to -\infty} \frac{x^2}{x^2 + 3} = 1
\]
Concavity and Inflection Points

Example

The first derivative of a certain function $f(x)$ is

$$f'(x) = x^2 - 2x - 8.$$

(a) Find intervals on which $f$ is increasing and decreasing.

(b) Find intervals on which the graph of $f$ is concave up and concave down.

(c) Find the $x$-coordinate of the relative extrema and inflection points of $f$.

\[\text{(a) Set } f' = 0. \text{ Solve } x^2 - 2x - 8 = 0\]

\[f'(-3) = -1(-7) > 0\]

\[f'(0) = 2(-4) < 0\]

\[x = -2 \quad x = 4 \leftarrow \text{ crit.}\]

\[\begin{array}{cccc}
-4 & -3 & -2 & 0 & 4 & 5 \\
\text{concave up} & \text{concave down} & \text{concave up} & \text{concave up} & \text{concave down} & \text{concave up}
\end{array}\]

\[f'(5) = (-7)(1) < 0\]
(b) \( f''(x) = 2x - 2 \)

Set \( f'' = 0 \). Solve: \( 2x - 2 = 0 \)

\( f''(0) = -2 < 0 \) \( \Rightarrow \) \( f''(2) = 2 > 0 \)

\( x = 1 \) \( \leftarrow \) possible inflection point.

\[ \frac{d^2y}{dx^2} \]

\[ \begin{array}{c|c|c|c}
\text{Value} & \text{Conc Down} & 0 & \text{Conc Up} \\
\hline
0 & 1 & 2 \\
\end{array} \]

(c) Local max at \((-2, ?)\)

Local min at \((4, ?)\)

Inflection point at \((1, ?)\)

We can only get x-coordinates.
The Second Derivative Test

Suppose $f''(x)$ exists on an open interval containing $x = c$ and that $f'(c) = 0$.

- If $f''(c) > 0$, then $f$ has a relative minimum at $x = c$.
- If $f''(c) < 0$, then $f$ has a relative maximum at $x = c$.

However, if $f''(c) = 0$ or if $f''(c)$ does not exist, the test is inconclusive and $f$ may have a relative maximum, a relative minimum, or no relative extremum at all at $x = c$. 
The Second Derivative Test

Example
Find the critical points of

\[ f(x) = x^3 + 3x^2 + 1 \]

and use the second derivative test to classify each critical point as a relative maximum or minimum.
e.g. \( f(x) = x^4 \)

Concavity: \( f'(x) = 4x^3 \) \( f''(x) = 12x^2 \)

Set \( f'' = 0 \). Solve: \( 12x^2 = 0 \) \( x = 0 \leftarrow \)

\[
\begin{array}{cccc}
-2 & -1 & 0 & 1 & 2 \\
\hline
- & + & + & 0 & +
\end{array}
\]

\( f''(1) = 12 > 0 \) \( f''(1) = 12 > 0 \)

\( \leftarrow \) concave up everywhere.
3.3. Curve Sketching

Vertical Asymptotes

The vertical line \( x = c \) is a vertical asymptote of the graph of \( f(x) \) if either

\[
\lim_{x \to c^-} f(x) = +\infty \quad \text{(or } -\infty) \]

or

\[
\lim_{x \to c^+} f(x) = +\infty \quad \text{(or } -\infty) \]

Check \( x \) where denominator \( = 0 \).
Vertical Asymptotes

Example
Determine all vertical asymptotes of the graph of

\[ g(x) = \frac{2x^2 + 2x}{x^2 - 3x - 4} \]

**Vertical asymptotes:**

\[ x^2 - 3x - 4 = 0 \]
\[ (x+1)(x-4) = 0 \]
\[ x = -1 \quad x = 4 \]

\[ \uparrow \]
\[ \text{vert. asymp.} \]

Check numerator:

\[ 2(4)^2 + 2(4) = 40 \neq 0, \]
\[ 2(-1)^2 + 2(-1) = 2 - 2 = 0 \]

\[ \lim_{x \to -1} \frac{2x^2 + 2x}{x^2 - 3x - 4} = \lim_{x \to -1} \frac{2x(x+1)}{(x+1)(x-4)} \]

\[ = \frac{2}{5} \]

\[ \text{Move work:} \]
\[ \text{Not a vert. asymp.} \]
\[ \text{(hole in graph)} \]
Horizontal Asymptotes

The horizontal line $y = b$ is a horizontal asymptote of the graph of $f(x)$ if

$$\lim_{{x \to -\infty}} f(x) = b$$

or

$$\lim_{{x \to +\infty}} f(x) = b$$
Horizontal Asymptotes

Example
Determine all horizontal asymptotes of the graph of

\[ g(x) = \frac{2x^2 + 2x}{x^2 - 3x - 4} \]

\[ \lim_{x \to \infty} \frac{2x^2 + 2x}{x^2 - 3x - 4} = \lim_{x \to \infty} \frac{2x^2}{x^2} = 2 \]
General Procedure for Sketching the Graph

Step 1. Find the domain of $f(x)$.
Step 2. Find and plot all intercepts.
Step 3. Determine all vertical and horizontal asymptotes and draw them.
Step 4. Find $f'(x)$ and determine the critical numbers and intervals of increase and decrease.
Step 5. Determine all relative extrema. Plot each relative maximum with a "cap" and each relative minimum with a "cup".
Step 6. Find $f''(x)$ and determine intervals of concavity and points of inflection. Plot inflection points with a "twist".
Step 7. Complete the sketch by joining the plotted points.
Curve Sketching

Example
Sketch the graph of \( f(x) = \frac{4x}{(x+1)^2}. \)

**Vertical asymptote at** \( x = -1 \)

**Horizontal asymptote at** \( y = 0 \)

\[
\lim_{x \to \infty} \frac{4x}{(x+1)^2} = \lim_{x \to \infty} \frac{4x}{x^2} = \lim_{x \to \infty} \frac{4}{x} = 0
\]

**Increase/Decrease.**

\[
f'(x) = \frac{(x+1)^2(4) - (4x)(2(x+1))}{(x+1)^4}
\]

\[
= \frac{(x+1)[4(x+1) - 8x]}{(x+1)^4} = \frac{-4(x+1)}{(x+1)^3}
\]

\[
= -\frac{4(x-1)}{(x+1)^3}
\]
Curve #: \( x = 1 \), \( x = -1 \)

\[
\begin{array}{cccccc}
  & f' & - & - & - & \bullet & + & + & + & 0 & - & - & - & \\
 & -2 & -1 & 0 & 1 & 2 \\
\end{array}
\]

\[
f'(-2) = \frac{(-4)(-3)}{(-1)^3} < 0 \quad f'(0) = \frac{(-4)(-1)}{(1)^3} > 0 \]

\[
f'(2) = \frac{(-4)(1)}{(3)^3} \leq 0 \quad \text{Local max at } (1, 1)
\]

Concavity:

\[
f''(x) = \frac{(x+1)^3(-4) - (-4(x-1))(2)(x+1)^2}{(x+1)^6}
\]

\[
= \frac{(x+1)^2[-4(x+1) - (-12(x-1))]}{(x+1)^6}
\]

\[
= \frac{-4x - 4 + 12x - 12}{(x+1)^4} = \frac{8x - 16}{(x+1)^4}
\]

\[
\frac{8(x-2)}{(x+1)^4}
\]
Possible inflection: \( x = 2 \) \( x = -1 \)

\[
\begin{array}{ccccccc}
-2 & -1 & 0 & 1 & 2 & 3 \\
\hline \\
\end{array}
\]

\[
f''(-2) = \frac{(8)(-4)}{(-1)^4} < 0 \quad f'(0) = \frac{(8)(-2)}{(1)^4} < 0 \\
\]

\[
f''(3) = \frac{(8)(1)}{(4)^4} > 0
\]

Inflection point at:
\[
(2, \frac{4(2)}{(2+1)^2}) = (2, \frac{8}{9})
\]

Graph showing increasing, decreasing, concave up, and concave down sections.

Horizontal asymptote: \( y = 0 \)
Curve Sketching

Example
Sketch the graph of $f(x) = \frac{x + 3}{x - 5}$. 
Curve Sketching

Example
Sketch the graph of \( f(x) = \frac{x + 1}{x^2 + x + 1} \).
3.4. Optimization

Absolute Maxima and Minima of a function
Let $f$ be a function defined on an interval $I$ containing the number $c$. Then

- $f(c)$ is the absolute maximum of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$.
- $f(c)$ is the absolute minimum of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.

Collectively, absolute maxima and minima are called absolute extrema.
Absolute Extrema on a Closed interval

How to Find the Absolute Extrema of a Continuous Function $f$ on $a \leq x \leq b$

Step 1. Find all critical numbers of $f$ in $a < x < b$.

Step 2. Compute $f(x)$ at the critical numbers found in step 1 and at the endpoints $x = a$ and $x = b$.

Step 3. The largest and smallest values found in step 2 are, respectively, the absolute maximum and absolute minimum values of $f(x)$ on $a \leq x \leq b$. 
Absolute Extrema on a Closed interval

Example
Find the absolute maximum and absolute minimum (if any) of

\[ f(x) = x^3 + 3x^2 + 1; \quad -3 \leq x \leq 2. \]

\[ f'(x) = 3x^2 + 6x \]

\[ 3x^2 + 6x = 0 \]

\[ 3x(x + 2) = 0 \]

\[ x = 0 \quad x = -2 \]

Critical points: \((-2, 5)\) \((0, 1)\)

2 more points:
\((-3, 1)\) \((2, 21)\)
Absolute Extrema on a Closed interval

Example
Find the absolute maximum and absolute minimum (if any) of

\[ f(t) = \frac{t^2}{t - 1}; \quad -2 \leq t \leq \frac{1}{2}. \]
Absolute Extrema on a general interval

Example
Find the absolute maximum and absolute minimum (if any) of

\[ f(u) = u + \frac{16}{u}; \quad u > 0. \]