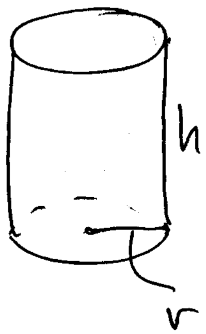
Exam 1 Monday 2-21
Coverage TBA
Wednesday 2-16 (Review)

Quiz 2
#14 p.57



Surface area = 120π

$V = \pi r^2 h$

A = surface area of can

$A = 2\pi r h$

$120\cancel{\pi} = 2\cancel{\pi} r h$

$60 = rh$

$h = \frac{60}{r}$ /

$V = \pi r^2 h = \pi r^2 \left(\frac{60}{r}\right) = 60\pi r$

Correct way:

Area of top and bottom

$A = 2\pi r h + 2\pi r^2$

$120\pi = 2\pi r h + 2\pi r^2$

$120\cancel{\pi} = 2\cancel{\pi}(rh + r^2)$

$60 = r(h+r) = rh + r^2$

$60 - r^2 = rh$

$h = \frac{60 - r^2}{r}$ /

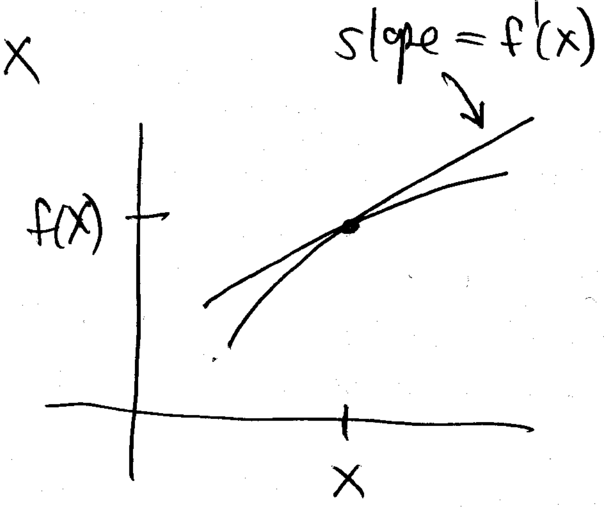
$V = \pi r^2 h$

$= \pi r^{\cancel{2}} \left(\frac{60 - r^2}{\cancel{r}}\right)$

$= \pi r (60 - r^2)$

Recap: (1) Derivative is limit of difference quotient: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x)$

(2) $f'(x)$ is the slope of tangent line to graph of f at x



(3) $f'(x)$ is the instantaneous rate of change of f with respect to x.

Instantaneous Rate of Change as a Derivative

The rate of change of $f(x)$ with respect to x when $x = c$ is given by $f'(c)$.

Example

A toy rocket rises vertically in such a way that t seconds after lift-off, it is

$$h(t) = -\frac{1}{2}t^2 + 20t$$

feet above ground.

feet/second

a. What is the (instantaneous) velocity of the rocket at lift-off? $h'(0)$

$h'(t)$ = velocity at time t.

b. What is its velocity after 10 seconds? $h'(10)$

Let's find $h'(t)$ for any t.

$$h'(t) = \lim_{s \to 0} \frac{h(t+s) - h(t)}{s} = \lim_{s \to 0} \frac{-\frac{1}{2}(t+s)^2 + 20(t+s) + \frac{1}{2}t^2 - 20t}{s}$$

$$= \lim_{s \to 0} \frac{-\frac{1}{2}(t^2 + 2st + s^2) + 20t + 20s + \frac{1}{2}t^2 - 20t}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{-\frac{1}{2}t^2} - st - \frac{1}{2}s^2 + \cancel{20t} + 20s + \cancel{\frac{1}{2}t^2} - \cancel{20t}}{s} =$$

$$= \lim_{s \to 0} \frac{\cancel{s}\left(-t - \frac{1}{2}s + 20\right)}{\cancel{s}} = -t + 20$$

$$h'(t) = -t + 20$$

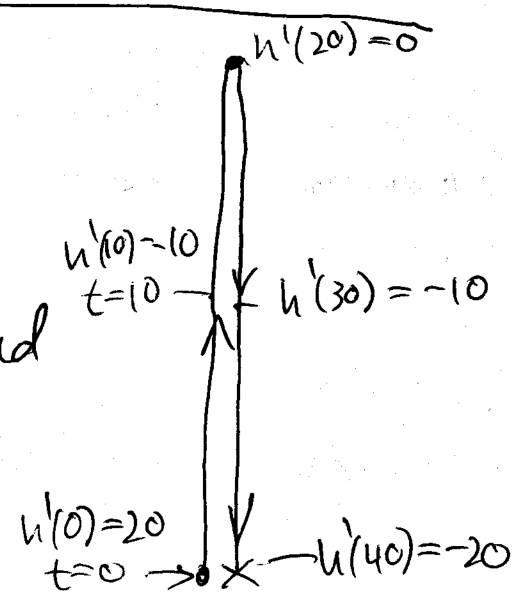
a. $h'(0) = 20$ feet/second

b. $h'(10) = -10 + 20 = 10$ feet/second

$h'(20) = 0$ feet/second.

$h'(30) = -10$ feet/second

$h'(40) = -20$ feet/second



$$h(40) = -\frac{1}{2}(40)^2 + 20(40)$$

$$= -\frac{1}{2}(40)(40) + (20)(40)$$

$$= -800 + 800 = 0.$$

Significance of the sign of $f'(x)$

If the function f is differentiable at $x = c$, then

$$f \text{ is } \textit{increasing} \text{ at } x = c \text{ if } f'(c) > 0$$

and

$$f \text{ is } \textit{decreasing} \text{ at } x = c \text{ if } f'(c) < 0$$

Example

c. At lift-off, is the rocket rising?

d. Is the rocket rising after 30 seconds?

Derivative Notation

The derivative $f'(x)$ of $y = f(x)$ is sometimes written as

$$\frac{dy}{dx} \text{ or } \frac{df}{dx}$$

In this notation, $f'(c)$ is written as

$$\left.\frac{dy}{dx}\right|_{x=c} \text{ or } \left.\frac{df}{dx}\right|_{x=c}$$

Example

Find the rate of change $\frac{dy}{dx}$ of $y = 5 - x^2$ at the point where $x = 2$.

$$\frac{dy}{dx} = -2x$$

$$\left.\frac{dy}{dx}\right|_{x=2} = -2(2) = -4$$

Differentiability and Continuity

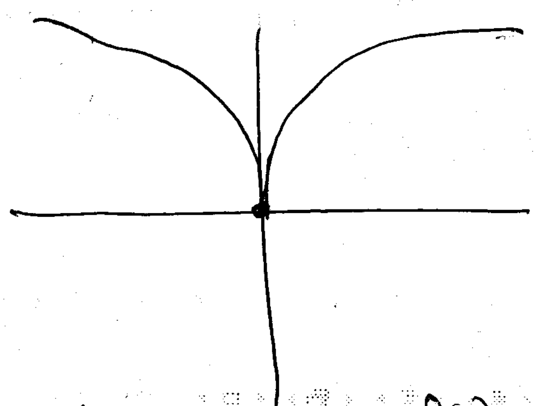
Continuity of a differentiable function

If the function $f(x)$ is differentiable at $x = c$, then it is also continuous at $x = c$. This means that for $f(x)$ to be differentiable at $x = c$ it must at least be continuous there, but *more is required.* There are functions that are continuous at a point but not differentiable there.

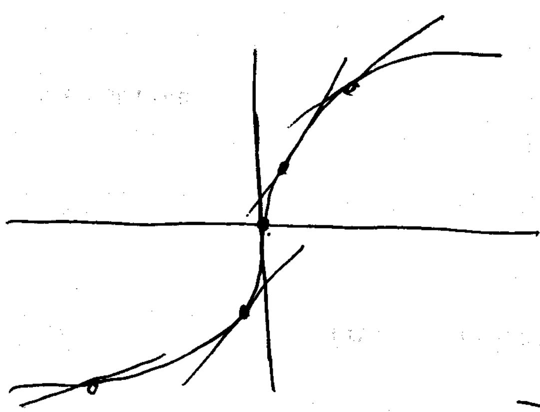
Examples of nondifferentiability

Each of the functions below is continuous at $x = 0$ but not differentiable at $x = 0$.

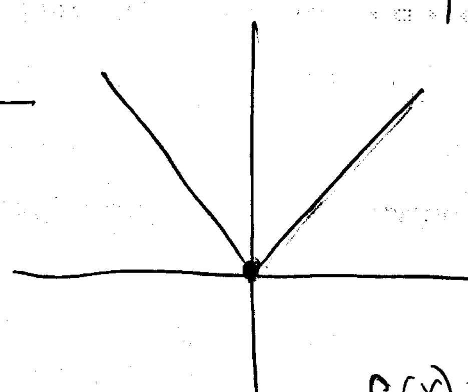
- Vertical tangent: $f(x) = x^{1/3}$
- Cusp: $f(x) = x^{2/3}$
- Corner: $f(x) = |x|$



$f(x) = x^{2/3}$
$= (x^2)^{1/3}$

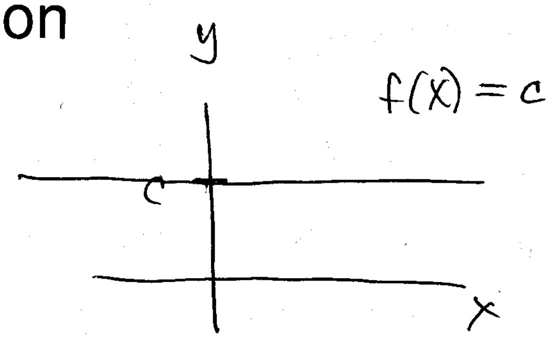


$f(x) = x^{1/3}$



$f(x) = |x|$

2.2. Techniques of Differentiation



The Constant Rule

For any constant c, $\quad \dfrac{d}{dx}[c] = 0$

The Power Rule

For any real number n, $\quad \dfrac{d}{dx}[x^n] = nx^{n-1}$

$$x^n \xrightarrow{\frac{d}{dx}} nx^{n-1}$$

Example

Differentiate the function $y = \sqrt{x^5}$.

$$y = \sqrt{x^5} = (x^5)^{1/2} = x^{5/2}$$

$$\frac{dy}{dx} = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{3/2} \quad //$$

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e.g. $\quad y = x^{-4} \qquad \dfrac{dy}{dx} = (-4)x^{-4-1} = -4x^{-5} \quad //$

e.g. $\quad y = \dfrac{1}{x^2} = x^{-2} \qquad \dfrac{dy}{dx} = -2x^{-3} \quad //$

The Constant Multiple Rule

If c is a constant and $f(x)$ is differentiable, then so is $cf(x)$ and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example

Differentiate the function $y = 2\sqrt[3]{x^4}$.

$$y = 2\sqrt[3]{x^4} = 2(x^4)^{1/3} = 2x^{4/3}$$

$$\frac{dy}{dx} = 2\frac{d}{dx}(x^{4/3}) = 2 \cdot \frac{4}{3}x^{\frac{4}{3}-1} = 2 \cdot \frac{4}{3}x^{1/3}$$

$$= \frac{8}{3}x^{1/3}$$

$$y = \frac{9}{\sqrt{t}} = 9t^{-1/2} \qquad \frac{dy}{dt} = \frac{d}{dt}(9t^{-1/2})$$

$$= 9\frac{d}{dt}(t^{-1/2}) = 9 \cdot \frac{-1}{2}t^{-3/2}$$

$$= -\frac{9}{2}t^{-3/2}$$

The Sum Rule

If $f(x)$ and $g(x)$ are differentiable, then so is their sum and

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Example

Differentiate the function $y = \frac{2}{x} - \frac{2}{x^2} + \frac{1}{3x^3}$.

$$y = 2x^{-1} - 2x^{-2} + \frac{1}{3}x^{-3}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^{-1} - 2x^{-2} + \frac{1}{3}x^{-3}\right)$$

$$= \frac{d}{dx}(2x^{-1}) + \frac{d}{dx}(-2x^{-2}) + \frac{d}{dx}\left(\frac{1}{3}x^{-3}\right)$$

$$= 2\frac{d}{dx}(x^{-1}) - 2\frac{d}{dx}(x^{-2}) + \frac{1}{3}\frac{d}{dx}(x^{-3})$$

$$= 2(-x^{-2}) - 2(-2x^{-3}) + \frac{1}{3}(-3x^{-4})$$

$$= -2x^{-2} + 4x^{-3} - x^{-4} \quad //$$

e.g. $y = x^2 - x$

$\frac{dy}{dx} = 2x - 1$

$x^{1} \xrightarrow{\frac{d}{dx}} 1x^0 = 1$

e.g. $y = 1 - 7x$

$\frac{dy}{dx} = 0 - 7 = -7$

$\frac{d}{dx}(1) = 0$

$\frac{d}{dx}(7x) = 7$

Differentiation of polynomials

Example

Differentiate the function $y = x^3(x^2 - 5x + 7)$.

$$y = x^3(x^2 - 5x + 7) = x^5 - 5x^4 + 7x^3$$

$$\frac{dy}{dx} = 5x^4 - 5 \cdot 4x^3 + 7 \cdot 3x^2$$

$$= 5x^4 - 20x^3 + 21x^2 \;//$$

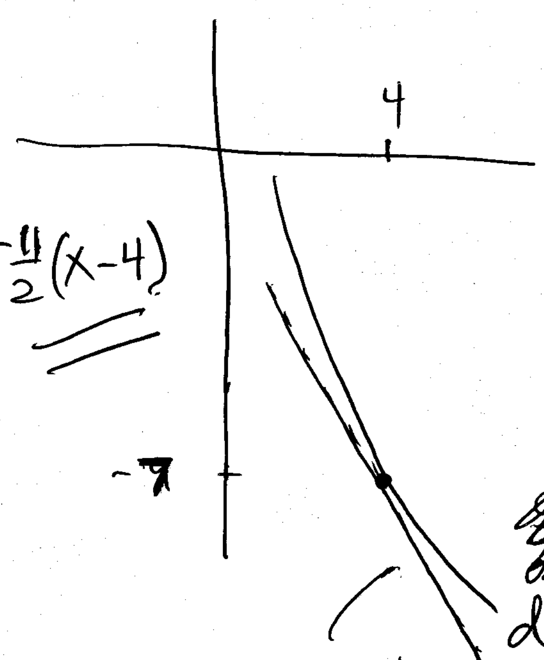
Equation of tangent lines

Example

Find the equation of the line that is tangent to the graph of the function $y = \sqrt{x^3} - x^2 + \frac{16}{x^2}$ at the point $(4, -7)$. //

$\boxed{\begin{aligned} &\text{Verify: } x = 4 \\ &y = \sqrt{(4)^3} - (4)^2 + \frac{16}{4^2} \\ &\quad = \sqrt{64} - 16 + 1 \\ &\quad = 8 - 16 + 1 = -7 \\ &\text{oops.} \end{aligned}}$

$y + 7 = \frac{-11}{2}(x - 4)$



$\text{slope} = -\frac{11}{2}$

$y = x^{3/2} - x^2 + 16x^{-2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} - 2x - 32x^{-3}$

$\left.\frac{dy}{dx}\right|_{x=4} = \frac{3}{2}(4)^{1/2} - 2(4) - 32(4)^{-3}$

$= \frac{3}{\cancel{2}}\cdot\cancel{2} - 8 - \frac{1}{2} = -\frac{11}{2}$ //

$\frac{3}{2} - 1 = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}$

Relative and Percentage Rate of Change

The *relative rate of change* of a quantity $Q(x)$ with respect to x is
$$\frac{Q'(x)}{Q(x)}$$
The corresponding *percentage rate of change* of $Q(x)$ with respect to x is
$$\frac{100Q'(x)}{Q(x)}$$

Relative and Percentage Rate of Change

Example

It is estimated that t years from now, the population of a certain town will be $P(t) = t^2 + 100t + 8,000$.

a. Express the percentage rate of change of the population as a function of t.

$P(t) = $ # people in town
$t = $ # years from now.

$$100\,\frac{P'(t)}{P(t)} = \frac{(2t+100)(100)}{t^2+100t+8000}$$

$P'(t) = 2t + 100 \leftarrow$ people per year.

b. What will happen to the percentage rate of change of the population in the long run?

Pct R.O.C. when $t = 10$?

$$\frac{(2(10)+100)(100)}{(10)^2+100(10)+8000} = \frac{120\cancel{00}}{91\cancel{00}}$$

$\cong 1.3\%$ per year.

Rectilinear Motion

Motion of an object along a line is called *rectilinear motion.*
If the *position* at time t of an object moving along a straight line is give by $s(t)$, the the object has

$$\text{velocity} \quad v(t) = s'(t) = \frac{dx}{dt}$$

and

$$\text{acceleration} \quad a(t) = v'(t) = \frac{dv}{dt}.$$

The object is	*moving to the right*	when $v(t) > 0$,
	moving to the left	when $v(t) < 0$, and
	stationary	when $v(t) = 0$.

Rectilinear Motion

Example

The position at time t of an object moving along a line is given by $s(t) = t^3 - 9t^2 + 15t + 25$.

a. Find the velocity of the object.

b. Find the total distance traveled by the object between $t = 0$ and $t = 6$.

c. Find the acceleration of the object and determine when the object is accelerating and decelerating between $t = 0$ and $t = 6$.